Brownian Motion

An Introduction to Stochastic Processes

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Solution Manual

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Problem 1.1 (Solution) a) We show the result for $\mathbb{R}^d$-valued random variables. Let $\xi, \eta \in \mathbb{R}^d$. By assumption,

$$\lim_{n\to\infty} \mathbb{E}\exp\left[i\left\langle\binom{\xi}{\eta}, \binom{X_n}{Y_n}\right\rangle\right] = \mathbb{E}\exp\left[i\left\langle\binom{\xi}{\eta}, \binom{X}{Y}\right\rangle\right]$$

$$\iff \lim_{n\to\infty} \mathbb{E}\exp\left[i\langle\xi, X_n\rangle + i\langle\eta, Y_n\rangle\right] = \mathbb{E}\exp\left[i\langle\xi, X\rangle + i\langle\eta, Y\rangle\right]$$

If we take $\xi = 0$ and $\eta = 0$, respectively, we see that

$$\lim_{n\to\infty} \mathbb{E}\exp\left[i\langle\eta, Y_n\rangle\right] = \mathbb{E}\exp\left[i\langle\eta, Y\rangle\right] \qquad \text{or} \qquad Y_n \xrightarrow{d} Y$$

$$\lim_{n\to\infty} \mathbb{E}\exp\left[i\langle\xi, X_n\rangle\right] = \mathbb{E}\exp\left[i\langle\xi, X\rangle\right] \qquad \text{or} \qquad X_n \xrightarrow{d} X.$$

Since $X_n \perp\!\!\!\perp Y_n$ we find

$$\begin{aligned}
\mathbb{E}\exp\left[i\langle\xi, X\rangle + i\langle\eta, Y\rangle\right] &= \lim_{n\to\infty} \mathbb{E}\exp\left[i\langle\xi, X_n\rangle + i\langle\eta, Y_n\rangle\right] \\
&= \lim_{n\to\infty} \mathbb{E}\exp\left[i\langle\xi, X_n\rangle\right] \mathbb{E}\exp\left[i\langle\eta, Y_n\rangle\right] \\
&= \lim_{n\to\infty} \mathbb{E}\exp\left[i\langle\xi, X_n\rangle\right] \lim_{n\to\infty} \mathbb{E}\exp\left[i\langle\eta, Y_n\rangle\right] \\
&= \mathbb{E}\exp\left[i\langle\xi, X\rangle\right] \mathbb{E}\exp\left[i\langle\eta, Y\rangle\right]
\end{aligned}$$

and this shows that $X \perp\!\!\!\perp Y$.

b) We have

$$\begin{aligned}
X_n = X + \frac{1}{n} &\xrightarrow[n\to\infty]{\text{almost surely}} X \implies X_n \xrightarrow{d} X \\
Y_n = 1 - X_n = 1 - \frac{1}{n} - X &\xrightarrow[n\to\infty]{\text{almost surely}} 1 - X \implies Y_n \xrightarrow{d} 1 - X \\
X_n + Y_n = 1 &\xrightarrow[n\to\infty]{\text{almost surely}} 1 \implies X_n + Y_n \xrightarrow{d} 1.
\end{aligned}$$

A simple direct calculation shows that $1 - X \sim \frac{1}{2}(\delta_0 + \delta_1) \sim Y$. Thus,

$$X_n \xrightarrow{d} X, \quad Y_n \xrightarrow{d} Y \sim 1 - X, \quad X_n + Y_n \xrightarrow{d} 1.$$

Assume that $(X_n, Y_n) \xrightarrow{d} (X, Y)$. Since $X \perp\!\!\!\perp Y$, we find for the distribution of $X + Y$:

$$X + Y \sim \tfrac{1}{2}(\delta_0 + \delta_1) * \tfrac{1}{2}(\delta_0 + \delta_1) = \tfrac{1}{4}(\delta_0 * \delta_0 + 2\delta_1 * \delta_0 + \delta_1 * \delta_1) = \tfrac{1}{4}(\delta_0 + 2\delta_1 + \delta_2).$$

Thus, $X + Y \not\sim \delta_0 \sim 1 = \lim_n (X_n + Y_n)$ and this shows that we cannot have that $(X_n, Y_n) \xrightarrow{d} (X, Y)$.

c) If $X_n \perp\!\!\!\perp Y_n$ and $X \perp\!\!\!\perp Y$, then we have $X_n + Y_n \xrightarrow{d} X + Y$: this follows since we have for all $\xi \in \mathbb{R}$:

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}\, e^{i\xi(X_n+Y_n)} &= \lim_{n\to\infty} \mathbb{E}\, e^{i\xi X_n}\, \mathbb{E}\, e^{i\xi Y_n}\\ &= \lim_{n\to\infty} \mathbb{E}\, e^{i\xi X_n} \lim_{n\to\infty} \mathbb{E}\, e^{i\xi Y_n}\\ &= \mathbb{E}\, e^{i\xi X}\; \mathbb{E}\, e^{i\xi Y}\\ &\overset{\text{a)}}{=} \mathbb{E}\left[e^{i\xi X} e^{i\xi Y}\right]\\ &= \mathbb{E}\, e^{i\xi(X+Y)}.\end{aligned}$$

A similar (even easier) argument works if $(X_n, Y_n) \xrightarrow{d} (X, Y)$. Then we have

$$f(x,y) := e^{i\xi(x+y)}$$

is bounded and continuous, i.e. we get directly

$$\lim_{n\to\infty} \mathbb{E}\, e^{i\xi(X_n+Y_n)} \lim_{n\to\infty} \mathbb{E}\, f(X_n, Y_n) = \mathbb{E}\, f(X,Y) = \mathbb{E}\, e^{i\xi(X+Y)}.$$

For a counterexample (if X_n and Y_n are not independent), see part b).

Notice that the independence and d-convergence of the sequences X_n, Y_n already implies $X \perp\!\!\!\perp Y$ and the d-convergence of the bivariate sequence (X_n, Y_n). This is a consequence of the following

Lemma. *Let $(X_n)_{n\geqslant 1}$ and $(Y_n)_{n\geqslant 1}$ be sequences of random variables (or random vectors) on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. If*

$$X_n \perp\!\!\!\perp Y_n \quad \text{for all } n \geqslant 1 \quad \text{and} \quad X_n \xrightarrow[n\to\infty]{d} X \quad \text{and} \quad Y_n \xrightarrow[n\to\infty]{d} Y,$$

then $(X_n, Y_n) \xrightarrow[n\to\infty]{d} (X, Y)$ and $X \perp\!\!\!\perp Y$.

Proof. Write $\phi_X, \phi_Y, \phi_{X,Y}$ for the characteristic functions of X, Y and the pair (X, Y). By assumption

$$\lim_{n\to\infty} \phi_{X_n}(\xi) = \lim_{n\to\infty} \mathbb{E}\, e^{i\xi X_n} = \mathbb{E}\, e^{i\xi X} = \phi_X(\xi).$$

A similar statement is true for Y_n and Y. For the pair we get, because of independence

$$\begin{aligned}\lim_{n\to\infty} \phi_{X_n,Y_n}(\xi,\eta) &= \lim_{n\to\infty} \mathbb{E}\, e^{i\xi X_n + i\eta Y_n}\\ &= \lim_{n\to\infty} \mathbb{E}\, e^{i\xi X_n}\, \mathbb{E}\, e^{i\eta Y_n}\\ &= \lim_{n\to\infty} \mathbb{E}\, e^{i\xi X_n} \lim_{n\to\infty} \mathbb{E}\, e^{i\eta Y_n}\\ &= \mathbb{E}\, e^{i\xi X}\, \mathbb{E}\, e^{i\eta Y}\\ &= \phi_X(\xi)\phi_Y(\eta).\end{aligned}$$

Thus, $\phi_{X_n,Y_n}(\xi,\eta) \to h(\xi,\eta) = \phi_X(\xi)\phi_Y(\eta)$. Since h is continuous at the origin $(\xi,\eta) = 0$ and $h(0,0) = 1$, we conclude from Lévy's continuity theorem that h is a (bivariate) characteristic function and that $(X_n,Y_n) \xrightarrow{d} (X,Y)$. Moreover,

$$h(\xi,\eta) = \phi_{X,Y}(\xi,\eta) = \phi_X(\xi)\phi_Y(\eta)$$

which shows that $X \perp\!\!\!\perp Y$. $\square$

Problem 1.2 (Solution) Using the elementary estimate

$$|e^{iz} - 1| = \left|\int_0^{iz} e^{\zeta}\,d\zeta\right| \leqslant \sup_{|y|\leqslant|z|} |e^{iy}|\,|z| = |z| \tag{*}$$

we see that the function $t \mapsto e^{i\langle\xi,t\rangle}$, $\xi, t \in \mathbb{R}^d$ is locally Lipschitz continuous:

$$\left|e^{i\langle\xi,t\rangle} - e^{i\langle\xi,s\rangle}\right| = \left|e^{i\langle\xi,t-s\rangle} - 1\right| \leqslant |\langle\xi,t-s\rangle| \leqslant |\xi|\cdot|t-s| \quad \text{for all} \quad \xi,t,s \in \mathbb{R}^d,$$

Thus,

$$\begin{aligned}\mathbb{E}\,e^{i\langle\xi,Y_n\rangle} &= \mathbb{E}\left[e^{i\langle\xi,Y_n-X_n\rangle}e^{i\langle\xi,X_n\rangle}\right]\\ &= \mathbb{E}\left[\left(e^{i\langle\xi,Y_n-X_n\rangle} - 1\right)e^{i\langle\xi,X_n\rangle}\right] + \mathbb{E}\,e^{i\langle\xi,X_n\rangle}.\end{aligned}$$

Since $\lim_{n\to\infty}\mathbb{E}\,e^{i\langle\xi,X_n\rangle} = \mathbb{E}\,e^{i\langle\xi,X\rangle}$, we are done if we can show that the first term in the last line of the displayed formula tends to zero. To see this, we use the Lipschitz continuity of the exponential function. Fix $\xi \in \mathbb{R}^d$.

$$\begin{aligned}&\left|\mathbb{E}\left[\left(e^{i\langle\xi,Y_n-X_n\rangle} - 1\right)e^{i\langle\xi,X_n\rangle}\right]\right|\\ &\quad\leqslant \mathbb{E}\left|\left(e^{i\langle\xi,Y_n-X_n\rangle} - 1\right)e^{i\langle\xi,X_n\rangle}\right|\\ &\quad= \mathbb{E}\left|e^{i\langle\xi,Y_n-X_n\rangle} - 1\right|\\ &\quad= \int_{|Y_n-X_n|\leqslant\delta}\left|e^{i\langle\xi,Y_n-X_n\rangle} - 1\right|d\mathbb{P} + \int_{|Y_n-X_n|>\delta}\left|e^{i\langle\xi,Y_n-X_n\rangle} - 1\right|d\mathbb{P}\\ &\quad\overset{(*)}{\leqslant} \delta\,|\xi| + \int_{|Y_n-X_n|>\delta} 2\,d\mathbb{P}\\ &\quad= \delta\,|\xi| + 2\,\mathbb{P}\left(|Y_n - X_n| > \delta\right)\\ &\quad\xrightarrow[n\to\infty]{} \delta\,|\xi| \xrightarrow[\delta\to 0]{} 0,\end{aligned}$$

where we used in the last step the fact that $X_n - Y_n \xrightarrow{\mathbb{P}} 0$.

Problem 1.3 (Solution) Recall that $Y_n \xrightarrow{d} Y$ with $Y = c$ a.s., i. e. where $Y \sim \delta_c$ for some constant $c \in \mathbb{R}$. Since the d-limit is trivial, this implies $Y_n \xrightarrow{\mathbb{P}} Y$. This means that both "is this still true"-questions can be answered in the affirmative.

We will show that $(X_n,Y_n) \xrightarrow{d} (X_n,c)$ holds – without assuming anything on the joint distribution of the random vector (X_n,Y_n), i.e. we do not make assumption on the correlation structure of X_n and Y_n. Since the maps $x \mapsto x + y$ and $x \mapsto x\cdot y$ are continuous, we see that

$$\lim_{n\to\infty}\mathbb{E}\,f(X_n,Y_n) = \mathbb{E}\,f(X,c) \quad \forall f \in C_b(\mathbb{R}\times\mathbb{R})$$

implies both

$$\lim_{n\to\infty} \mathbb{E}\, g(X_n Y_n) = \mathbb{E}\, g(Xc) \quad \forall g \in C_b(\mathbb{R})$$

and

$$\lim_{n\to\infty} \mathbb{E}\, h(X_n + Y_n) = \mathbb{E}\, h(X + c) \quad \forall h \in C_b(\mathbb{R}).$$

This proves (a) and (b).

In order to show that (X_n, Y_n) converges in distribution, we use Lévy's characterization of distributional convergence, i.e. the pointwise convergence of the characteristic functions. This means that we take $f(x,y) = e^{i(\xi x + \eta y)}$ for any $\xi, \eta \in \mathbb{R}$:

$$\begin{aligned}
\left|\mathbb{E}\, e^{i(\xi X_n + \eta Y_n)} - \mathbb{E}\, e^{i(\xi X + \eta c)}\right| &\leqslant \left|\mathbb{E}\, e^{i(\xi X_n + \eta Y_n)} - \mathbb{E}\, e^{i(\xi X_n + \eta c)}\right| + \left|\mathbb{E}\, e^{i(\xi X_n + \eta c)} - \mathbb{E}\, e^{i(\xi X + \eta c)}\right| \\
&\leqslant \mathbb{E}\left|e^{i(\xi X_n + \eta Y_n)} - \mathbb{E}\, e^{i(\xi X_n + \eta c)}\right| + \left|\mathbb{E}\, e^{i(\xi X_n + \eta c)} - \mathbb{E}\, e^{i(\xi X + \eta c)}\right| \\
&\leqslant \mathbb{E}\left|e^{i\eta Y_n} - e^{i\eta c}\right| + \left|\mathbb{E}\, e^{i\xi X_n} - \mathbb{E}\, e^{i\xi X}\right|.
\end{aligned}$$

The second expression on the right-hand side converges to zero as $X_n \xrightarrow{d} X$. For fixed η we have that $y \mapsto e^{i\eta y}$ is uniformly continuous. Therefore, the first expression on the right-hand side becomes, with any $\epsilon > 0$ and a suitable choice of $\delta = \delta(\epsilon) > 0$

$$\begin{aligned}
\mathbb{E}\left|e^{i\eta Y_n} - e^{i\eta c}\right| &= \mathbb{E}\left[\left|e^{i\eta Y_n} - e^{i\eta c}\right| \mathbb{1}_{\{|Y_n - c| > \delta\}}\right] + \mathbb{E}\left[\left|e^{i\eta Y_n} - e^{i\eta c}\right| \mathbb{1}_{\{|Y_n - c| \leqslant \delta\}}\right] \\
&\leqslant 2\,\mathbb{E}\left[\mathbb{1}_{\{|Y_n - c| > \delta\}}\right] + \mathbb{E}\left[\epsilon \mathbb{1}_{\{|Y_n - c| \leqslant \delta\}}\right] \\
&\leqslant 2\,\mathbb{P}(|Y_n - c| > \delta) + \epsilon \\
&\xrightarrow[n\to\infty]{\mathbb{P}\text{-convergence as } \delta,\epsilon \text{ are fixed}} \epsilon \xrightarrow[\epsilon\downarrow 0]{} 0.
\end{aligned}$$

Remark. The direct approach to (a) is possible but relatively ugly. Part (b) has a relatively simple direct proof:

Fix $\xi \in \mathbb{R}$.

$$\mathbb{E}\, e^{i\xi(X_n + Y_n)} - \mathbb{E}\, e^{i\xi X} = \left(\mathbb{E}\, e^{i\xi(X_n+Y_n)} - \mathbb{E}\, e^{i\xi X_n}\right) + \underbrace{\left(\mathbb{E}\, e^{i\xi X_n} - \mathbb{E}\, e^{i\xi X}\right)}_{\xrightarrow[n\to\infty]{} 0 \text{ by } d\text{-convergence}}.$$

For the first term on the right we find with the uniform-continuity argument from Problem 1.1.2 and any $\epsilon > 0$ and suitable $\delta = \delta(\epsilon, \xi)$ that

$$\begin{aligned}
\left|\mathbb{E}\, e^{i\xi(X_n+Y_n)} - \mathbb{E}\, e^{i\xi X_n}\right| &\leqslant \mathbb{E}\left|e^{i\xi X_n}\left(e^{i\xi Y_n} - 1\right)\right| \\
&= \mathbb{E}\left|e^{i\xi Y_n} - 1\right| \\
&\leqslant \epsilon + \mathbb{P}\left(|Y_n| > \delta\right) \\
&\xrightarrow[n\to\infty]{\epsilon \text{ fixed}} \epsilon \xrightarrow[\epsilon\to 0]{} 0
\end{aligned}$$

where we use $\mathbb{P}$-convergence in the penultimate step.

Problem 1.4 (Solution) Let $\xi, \eta \in \mathbb{R}$ and note that $f(x) = e^{i\xi x}$ and $g(y) = e^{i\eta y}$ are bounded and continuous functions. Thus we get

$$\begin{aligned}
\mathbb{E}\, e^{i\left\langle\binom{\xi}{\eta}, \binom{X}{Y}\right\rangle} &= \mathbb{E}\, e^{i\xi X} e^{i\eta Y} \\
&= \mathbb{E}\, f(X) g(Y) \\
&= \lim_{n\to\infty} \mathbb{E}\, f(X_n) g(Y) \\
&= \lim_{n\to\infty} \mathbb{E}\, e^{i\xi X_n} e^{i\eta Y} \\
&= \lim_{n\to\infty} \mathbb{E}\, e^{i\left\langle\binom{\xi}{\eta}, \binom{X_n}{Y}\right\rangle}
\end{aligned}$$

and we see that $(X_n, Y) \xrightarrow{d} (X, Y)$.

Assume now that $X = \phi(Y)$ for some Borel function ϕ. Let $f \in \mathcal{C}_b$ and pick $g := f \circ \phi$. Clearly, $f \circ \phi \in \mathcal{B}_b$ and we get

$$\begin{aligned}
\mathbb{E}\, f(X_n) f(X) &= \mathbb{E}\, f(X_n) f(\phi(Y)) \\
&= \mathbb{E}\, f(X_n) g(Y) \\
&\xrightarrow[n\to\infty]{} \mathbb{E}\, f(X) g(Y) \\
&= \mathbb{E}\, f(X) f(X) \\
&= \mathbb{E}\, f^2(X).
\end{aligned}$$

Now observe that $f \in \mathcal{C}_b \implies f^2 \in \mathcal{C}_b$ and $g \equiv 1 \in \mathcal{B}_b$. By assumption

$$\mathbb{E}\, f^2(X_n) \xrightarrow[n\to\infty]{} \mathbb{E}\, f^2(X).$$

Thus,

$$\begin{aligned}
\mathbb{E}\left(|f(X) - f(X_n)|^2\right) &= \mathbb{E}\, f^2(X_n) - 2\,\mathbb{E}\, f(X_n) f(X) + \mathbb{E}\, f^2(X) \\
&\xrightarrow[n\to\infty]{} \mathbb{E}\, f^2(X) - 2\,\mathbb{E}\, f(X) f(X) + \mathbb{E}\, f^2(X) = 0,
\end{aligned}$$

i.e. $f(X_n) \xrightarrow{L^2} f(X)$.

Now fix $\epsilon > 0$ and $R > 0$ and set $f(x) = -R \vee x \wedge R$. Clearly, $f \in \mathcal{C}_b$. Then

$$\begin{aligned}
&\mathbb{P}(|X_n - X| > \epsilon) \\
&\quad \leqslant \mathbb{P}(|X_n - X| > \epsilon,\ |X| \leqslant R,\ |X_n| \leqslant R) + \mathbb{P}(|X| \geqslant R) + \mathbb{P}(|X_n| \geqslant R) \\
&\quad = \mathbb{P}(|f(X_n) - f(X)| > \epsilon,\ |X| \leqslant R,\ |X_n| \leqslant R) + \mathbb{P}(|X| \geqslant R) + \mathbb{P}(|f(X_n)| \geqslant R) \\
&\quad \leqslant \mathbb{P}(|f(X_n) - f(X)| > \epsilon) + \mathbb{P}(|X| \geqslant R) + \mathbb{P}(|f(X_n)| \geqslant R) \\
&\quad \leqslant \mathbb{P}(|f(X_n) - f(X)| > \epsilon) + \mathbb{P}(|X| \geqslant R) + \mathbb{P}(|f(X)| \geqslant R/2) + \mathbb{P}(|f(X_n) - f(X)| \geqslant R/2)
\end{aligned}$$

where we used that $\{|f(X_n)| \geqslant R\} \subset \{|f(X)| \geqslant R/2\} \cup \{|f(X_n) - f(X)| \geqslant R/2\}$ because of the triangle inequality: $|f(X_n)| \leqslant |f(X)| + |f(X) - f(X_n)|$

$$= \mathbb{P}(|f(X_n) - f(X)| > \epsilon) + \mathbb{P}(|X| \geqslant R/2) + \mathbb{P}(|X| \geqslant R/2) + \mathbb{P}(|f(X_n) - f(X)| \geqslant R/2)$$

$$
\begin{aligned}
&= \mathbb{P}(|f(X_n) - f(X)| > \epsilon) + 2\,\mathbb{P}(|X| \geqslant R/2) + \mathbb{P}(|f(X_n) - f(X)| \geqslant R/2) \\
&\leqslant \left(\frac{1}{\epsilon^2} + \frac{4}{R^2}\right) \mathbb{E}\left(|f(X) - f(X_n)|^2\right) + 2\,\mathbb{P}(|X| \geqslant R/2) \\
&\xrightarrow[n\to\infty]{\epsilon, R \text{ fixed and } f = f_R \in \mathcal{C}_b} 2\,\mathbb{P}(|X| \geqslant R/2) \xrightarrow[R\to\infty]{X \text{ is a.s. } \mathbb{R}\text{-valued}} 0.
\end{aligned}
$$

Problem 1.5 (Solution) Note that $\mathbb{E}\,\delta_j = 0$ and $\mathbb{V}\,\delta_j = \mathbb{E}\,\delta_j^2 = 1$. Thus, $\mathbb{E}\,S_{\lfloor nt\rfloor} = 0$ and $\mathbb{V}\,S_{\lfloor nt\rfloor} = \lfloor nt\rfloor$.

a) We have, by the central limit theorem (CLT)

$$
\frac{S_{\lfloor nt\rfloor}}{\sqrt{n}} = \frac{\sqrt{\lfloor nt\rfloor}}{\sqrt{n}}\,\frac{S_{\lfloor nt\rfloor}}{\sqrt{\lfloor nt\rfloor}} \xrightarrow[n\to\infty]{\text{CLT}} \sqrt{t}\,G_1
$$

where $G_1 \sim \mathsf{N}(0,1)$, hence $G_t := \sqrt{t}\,G_1 \sim N(0,t)$.

b) Let $s < t$. Since the δ_j are iid, we have, $S_{\lfloor nt\rfloor} - S_{\lfloor ns\rfloor} \sim S_{\lfloor nt\rfloor - \lfloor ns\rfloor}$, and by the central limit theorem (CLT)

$$
\frac{S_{\lfloor nt\rfloor - \lfloor ns\rfloor}}{\sqrt{n}} = \frac{\sqrt{\lfloor nt\rfloor - \lfloor ns\rfloor}}{\sqrt{n}}\,\frac{S_{\lfloor nt\rfloor - \lfloor ns\rfloor}}{\sqrt{\lfloor nt\rfloor - \lfloor ns\rfloor}} \xrightarrow[n\to\infty]{\text{CLT}} \sqrt{t-s}\,G_1 \sim G_{t-s}.
$$

If we know that the bivariate random variable $(S_{\lfloor ns\rfloor}, S_{\lfloor nt\rfloor} - S_{\lfloor ns\rfloor})$ converges in distribution, we do get $G_t \sim G_s + G_{t-s}$ because of Problem 1.1. But this follows again from the lemma which we prove in part d). This lemma shows that the limit has independent coordinates, see also part c). This is as close as we can come to $G_t - G_s \sim G_{t-s}$, unless we have a realization of ALL the G_t on a good space. It is Brownian motion which will achieve just this.

c) We know that the entries of the vector $(X^n_{t_m} - X^n_{t_{m-1}}, \dots, X^n_{t_2} - X^n_{t_1}, X^n_{t_1})$ are independent (they depend on different blocks of the δ_j and the δ_j are iid) and, by the one-dimensional argument of b) we see that

$$
X^n_{t_k} - X^n_{t_{k-1}} \xrightarrow[n\to\infty]{d} \sqrt{t_k - t_{k-1}}\,G_1^k \sim G^k_{t_k - t_{k-1}} \qquad \text{for all} \quad k = 1, \dots, m
$$

where the G_1^k, $k = 1, \dots, m$ are standard normal random vectors.

By the lemma in part d) we even see that

$$
(X^n_{t_m} - X^n_{t_{m-1}}, \dots, X^n_{t_2} - X^n_{t_1}, X^n_{t_1}) \xrightarrow[n\to\infty]{d} (\sqrt{t_1}G_1^1, \dots, \sqrt{t_m - t_{m-1}}G_1^m)
$$

and the G_1^k, $k = 1, \dots, m$ are independent. Thus, by the second assertion of part b)

$$
(\sqrt{t_1}G_1^1, \dots, \sqrt{t_m - t_{m-1}}G_1^m) \sim (G^1_{t_1}, \dots, G^m_{t_m - t_{m-1}}) \sim (G_{t_1}, \dots, G_{t_m} - G_{t_{m-1}}).
$$

d) We have the following

Lemma. *Let $(X_n)_{n\geqslant 1}$ and $(Y_n)_{n\geqslant 1}$ be sequences of random variables (or random vectors) on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. If*

$$
X_n \perp\!\!\!\perp Y_n \quad \textit{for all} \quad n \geqslant 1 \quad \textit{and} \quad X_n \xrightarrow[n\to\infty]{d} X \quad \textit{and} \quad Y_n \xrightarrow[n\to\infty]{d} Y,
$$

then $(X_n, Y_n) \xrightarrow[n\to\infty]{d} (X, Y)$ and $X \perp\!\!\!\perp Y$ (for suitable versions of the rv's).

Proof. Write $\phi_X, \phi_Y, \phi_{X,Y}$ for the characteristic functions of X, Y and the pair (X,Y). By assumption

$$\lim_{n\to\infty} \phi_{X_n}(\xi) = \lim_{n\to\infty} \mathbb{E}\, e^{i\xi X_n} = \mathbb{E}\, e^{i\xi X} = \phi_X(\xi).$$

A similar statement is true for Y_n and Y. For the pair we get, because of independence

$$\begin{aligned}
\lim_{n\to\infty} \phi_{X_n,Y_n}(\xi,\eta) &= \lim_{n\to\infty} \mathbb{E}\, e^{i\xi X_n + i\eta Y_n} \\
&= \lim_{n\to\infty} \mathbb{E}\, e^{i\xi X_n}\, \mathbb{E}\, e^{i\eta Y_n} \\
&= \lim_{n\to\infty} \mathbb{E}\, e^{i\xi X_n} \lim_{n\to\infty} \mathbb{E}\, e^{i\eta Y_n} \\
&= \mathbb{E}\, e^{i\xi X}\, \mathbb{E}\, e^{i\eta Y} \\
&= \phi_X(\xi)\phi_Y(\eta).
\end{aligned}$$

Thus, $\phi_{X_n,Y_n}(\xi,\eta) \to h(\xi,\eta) = \phi_X(\xi)\phi_Y(\eta)$. Since h is continuous at the origin $(\xi,\eta) = 0$ and $h(0,0) = 1$, we conclude from Lévy's continuity theorem that h is a (bivariate) characteristic function and that $(X_n, Y_n) \xrightarrow{d} (X,Y)$. Moreover,

$$h(\xi,\eta) = \phi_{X,Y}(\xi,\eta) = \phi_X(\xi)\phi_Y(\eta)$$

which shows that $X \perp\!\!\!\perp Y$. $\square$

Problem 1.6 (Solution) Necessity is clear. For sufficiency write

$$\frac{B(t)-B(s)}{\sqrt{t-s}} = \frac{1}{\sqrt{2}}\left(\frac{B(t)-B(\frac{s+t}{2})}{\sqrt{\frac{t-s}{2}}} + \frac{B(\frac{s+t}{2})-B(s)}{\sqrt{\frac{t-s}{2}}}\right) =: \frac{1}{\sqrt{2}}\,(X+Y)\,.$$

By assumption $X \sim Y$, $X \perp\!\!\!\perp Y$ and $X \sim \frac{1}{\sqrt{2}}(X+Y)$. This is already enough to guarantee that $X \sim \mathsf{N}(0,1)$, cf. Rényi [8, Chapter VI.5, Theorem 2, pp. 324–325].

<u>Alternative Solution:</u> Fix $s<t$ and define $t_j := s + \frac{j}{n}(t-s)$ for $j = 0,\dots,n$. Then

$$B_t - B_s = \sqrt{t_j - t_{j-1}} \sum_{j=1}^n \frac{B_{t_j} - B_{t_{j-1}}}{\sqrt{t_j - t_{j-1}}} = \sqrt{\frac{t-s}{n}} \sum_{j=1}^n \underbrace{\frac{B_{t_j} - B_{t_{j-1}}}{\sqrt{t_j - t_{j-1}}}}_{=:G_j^n}$$

By assumption, the random variables $(G_j^n)_{j,n}$ are identically distributed (for all j,n) and independent (in j). Moreover, $\mathbb{E}(G_j^n) = 0$ and $\mathbb{V}(G_j^n) = 1$. Applying the central limit theorem (for triangular arrays) we obtain

$$\frac{1}{\sqrt{n}} \sum_{j=1}^n G_j^n \xrightarrow{d} G_1$$

where $G_1 \sim \mathsf{N}(0,1)$. Thus, $B_t - B_s \sim \mathsf{N}(0, t-s)$.

2 Brownian motion as a Gaussian process

Problem 2.1 (Solution) Let us check first that $f(u,v) := g(u)g(v)(1-\sin u \sin v)$ is indeed a probability density. Clearly, $f(u,v) \geqslant 0$. Since $g(u) = (2\pi)-1/2\, e^{-u^2/2}$ is even and $\sin u$ is odd, we get

$$\iint f(u,v)\,du\,dv = \int g(u)\,du \int g(v)\,dv - \int g(u)\sin u\,du \int g(v)\sin v\,dv = 1 - 0.$$

Moreover, the density $f_U(u)$ of U is

$$f_U(u) = \int f(u,v)\,dv = g(u)\int g(v)\,dv - g(u)\sin u \int g(v)\sin v\,dv = g(u).$$

This, and a analogous argument show that $U, V \sim \mathsf{N}(0,1)$.

Let us show that (U,V) is not a normal random variable. Assume that (U,V) is normal, then $U+V \sim \mathsf{N}(0,\sigma^2)$, i.e.

$$\mathbb{E}\, e^{i\xi(U+V)} = e^{-\frac{1}{2}\xi^2\sigma^2}. \tag{*}$$

On the other hand we calculate with $f(u,v)$ that

$$\begin{aligned}
\mathbb{E}\, e^{i\xi(U+V)} &= \iint e^{i\xi u + i\xi v} f(u,v)\,du\,dv \\
&= \Big(\int e^{i\xi u} g(u)\,du\Big)^2 - \Big(\int e^{i\xi u} g(u)\sin u\,du\Big)^2 \\
&= e^{-\xi^2} - \Big(\frac{1}{2i}\int e^{i\xi u}(e^{iu} - e^{-iu})g(u)\,du\Big)^2 \\
&= e^{-\xi^2} - \Big(\frac{1}{2i}\int \big(e^{i(\xi+1)u} - e^{i(\xi-1)u}\big)g(u)\,du\Big)^2 \\
&= e^{-\xi^2} - \Big(\frac{1}{2i}\Big(e^{-\frac{1}{2}(\xi+1)^2} - e^{-\frac{1}{2}(\xi-1)^2}\Big)\Big)^2 \\
&= e^{-\xi^2} + \frac{1}{4}\Big(e^{-\frac{1}{2}(\xi+1)^2} - e^{-\frac{1}{2}(\xi-1)^2}\Big)^2 \\
&= e^{-\xi^2} + \frac{1}{4}\, e^{-1} e^{-\xi^2}\big(e^{-\xi} - e^{\xi}\big)^2,
\end{aligned}$$

and this contradicts (*).

Problem 2.2 (Solution) Let $(\xi_1,\dots,\xi_n) \neq (0,\dots,0)$ and set $t_0 = 0$. Then we find from (2.12)

$$\sum_{j=1}^{n}\sum_{k=1}^{n}(t_j \wedge t_k)\,\xi_j\xi_k = \sum_{j=1}^{n}\underbrace{(t_j - t_{j-1})}_{>0}(\xi_j + \dots + \xi_n)^2 \geqslant 0. \tag{2.1}$$

Equality (= 0) occurs if, and only if, $(\xi_j + \dots + \xi_n)^2 = 0$ for all $j = 1,\dots,n$. This implies that $\xi_1 = \ldots = \xi_n = 0$.

Abstract alternative: Let $(X_t)_{t\in I}$ be a real-valued stochastic process which has a second moment (such that the covariance is defined!), set $\mu_t = \mathbb{E} X_t$. For any finite set $S \subset I$ we pick $\lambda_s \in \mathbb{C}$, $s \in S$. Then

$$\begin{aligned}
\sum_{s,t\in S} \operatorname{Cov}(X_s, X_t)\lambda_s\bar{\lambda}_t &= \sum_{s,t\in S} \mathbb{E}\big((X_s-\mu_s)(X_t-\mu_t)\big)\lambda_s\bar{\lambda}_t \\
&= \mathbb{E}\left(\sum_{s,t\in S}(X_s-\mu_s)\lambda_s\overline{(X_t-\mu_t)\lambda_t}\right) \\
&= \mathbb{E}\left(\sum_{s\in S}(X_s-\mu_s)\lambda_s\overline{\sum_{t\in S}(X_t-\mu_t)\lambda_t}\right) \\
&= \mathbb{E}\left(\left|\sum_{s\in S}(X_s-\mu_s)\lambda_s\right|^2\right) \geqslant 0.
\end{aligned}$$

Remark: Note that this alternative does not prove that the covariance is *strictly* positive definite. A standard counterexample is to take $X_s \equiv X$.

Problem 2.3 (Solution) These are direct & straightforward calculations.

Problem 2.4 (Solution) Let $e_i = (\underbrace{0,\dots,0,1}_{i},0\dots) \in \mathbb{R}^n$ be the ith standard unit vector. Then

$$a_{ii} = \langle Ae_i, e_i\rangle = \langle Be_i, e_i\rangle = b_{ii}.$$

Moreover, for $i \neq j$, we get by the symmetry of A and B

$$\langle A(e_i+e_j), e_i+e_j\rangle = a_{ii} + a_{jj} + 2b_{ij}$$

and

$$\langle B(e_i+e_j), e_i+e_j\rangle = b_{ii} + b_{jj} + 2b_{ij}$$

which shows that $a_{ij} = b_{ij}$. Thus, $A = B$.

We have

Let $A, B \in \mathbb{R}^{n\times n}$ be symmetric matrices. If $\langle Ax, x\rangle = \langle Bx, x\rangle$ for all $x \in \mathbb{R}^n$, then $A = B$.

Problem 2.5 (Solution) a) $X_t = 2B_{t/4}$ is a BM^1: scaling property with $c = 1/4$, cf. 2.12.

b) $Y_t = B_{2t} - B_t$ is not a BM^1, the independent increments is clearly violated:

$$\begin{aligned}
\mathbb{E}(Y_{2t}-Y_t)Y_t &= \mathbb{E}(Y_{2t}Y_t) - \mathbb{E} Y_t^2 \\
&= \mathbb{E}(B_{4t}-B_{2t})(B_{2t}-B_t) - \mathbb{E}(B_{2t}-B_t)^2 \\
&\overset{\text{(B1)}}{=} \mathbb{E}(B_{4t}-B_{2t})\,\mathbb{E}(B_{2t}-B_t) - \mathbb{E}(B_{2t}-B_t)^2 \\
&\overset{\text{(B1)}}{=} -\mathbb{E}(B_t^2) = -t \neq 0.
\end{aligned}$$

c) $Z_t = \sqrt{t}B_1$ is not a BM^1, the independent increments property is violated:

$$\mathbb{E}(Z_t - Z_s)Z_s = (\sqrt{t}-\sqrt{s})\sqrt{s}\,\mathbb{E} B_1^2 = (\sqrt{t}-\sqrt{s})\sqrt{s} \neq 0.$$

Problem 2.6 (Solution) We use formula (2.10b).

a) $f_{B(s),B(t)}(x,y) = \dfrac{1}{2\pi\sqrt{s(t-s)}} \exp\left[-\dfrac{1}{2}\left(\dfrac{x^2}{s} + \dfrac{(y-x)^2}{t-s}\right)\right]$.

b) Denote by $f_{B(1)}$ the density of $B(1)$. Then we have

$$
\begin{aligned}
&f_{B(s),B(t)\,|\,B(1)}(x,y|B(1)=z)\\
&= \frac{f_{B(s),B(t),B(1)}(x,y,z)}{f_{B(1)}(z)}\\
&= \frac{1}{(2\pi)^{3/2}\sqrt{s(t-s)(1-t)}} \exp\left[-\frac{1}{2}\left(\frac{x^2}{s} + \frac{(y-x)^2}{t-s} + \frac{(z-y)^2}{1-t}\right)\right] (2\pi)^{1/2} \exp\left[\frac{z^2}{2}\right].
\end{aligned}
$$

Thus,

$$
f_{B(s),B(t)|B(1)}(x,y\,|\,B(1)=0) = \frac{1}{2\pi\sqrt{s(t-s)(1-t)}} \exp\left[-\frac{1}{2}\left(\frac{x^2}{s} + \frac{(y-x)^2}{t-s} + \frac{y^2}{1-t}\right)\right].
$$

Note that

$$
\frac{x^2}{s} + \frac{(y-x)^2}{t-s} + \frac{y^2}{1-t} = \frac{t}{s(t-s)}\left(x - \frac{s}{t}\,y\right)^2 + \frac{y^2}{t} + \frac{y^2}{1-t} = \frac{t}{s(t-s)}\left(x - \frac{s}{t}\,y\right)^2 + \frac{y^2}{t(1-t)}.
$$

Therefore,

$$
\begin{aligned}
&\mathbb{E}(B(s)B(t)\,|\,B(1)=0)\\
&= \iint xy f_{B(s),B(t)|B(1)}(x,y\,|\,B(1)=0)\,dx\,dy\\
&= \frac{1}{2\pi\sqrt{s(t-s)(1-t)}} \int_{y=-\infty}^{\infty} y \exp\left[-\frac{1}{2}\,\frac{y^2}{t(1-t)}\right] \times\\
&\qquad \times \underbrace{\int_{x=-\infty}^{\infty} x \exp\left[-\frac{1}{2}\,\frac{t}{s(t-s)}\left(x - \frac{s}{t}\,y\right)^2\right] dx}_{=\frac{\sqrt{s(t-s)}}{\sqrt{t}}\sqrt{2\pi}\,\frac{s}{t}\,y}\,dy\\
&= \frac{1}{\sqrt{2\pi}\sqrt{t(1-t)}} \int_{y=-\infty}^{\infty} y^2\,\frac{s}{t}\,\exp\left[-\frac{1}{2}\,\frac{y^2}{t(1-t)}\right] dy\\
&= \frac{s}{t}\,t(1-t) = s(1-t).
\end{aligned}
$$

c) In analogy to part b) we get

$$
\begin{aligned}
&f_{B(t_2),B(t_3)|B(t_1),B(t_4)}(x,y\,|\,B(t_1)=u, B(t_4)=z)\\
&= \frac{f_{B(t_1),B(t_2),B(t_3),B(t_4)}(u,x,y,z)}{f_{B(t_1),B(t_4)}(u,z)}\\
&= \frac{1}{2\pi}\left[\frac{t_1(t_4-t_1)}{t_1(t_2-t_1)(t_3-t_2)(t_4-t_3)}\right]^{\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\frac{u^2}{t_1} + \frac{(x-u)^2}{t_2-t_1} + \frac{(y-x)^2}{t_3-t_2} + \frac{(z-y)^2}{t_4-t_3}\right)\right]\times\\
&\qquad \times \exp\left[\frac{1}{2}\left(\frac{u^2}{t_1} + \frac{(z-u)^2}{t_4-t_1}\right)\right].
\end{aligned}
$$

Thus,

$$
f_{B(t_2),B(t_3)|B(t_1),B(t_4)}(x,y\,|\,B(t_1)=B(t_4)=0)
$$

$$= \frac{1}{2\pi}\left[\frac{t_1(t_4-t_1)}{t_1(t_2-t_1)(t_3-t_2)(t_4-t_3)}\right]^{\frac12}\exp\left[-\frac12\left(\frac{x^2}{t_2-t_1}+\frac{(y-x)^2}{t_3-t_2}+\frac{y^2}{t_4-t_3}\right)\right].$$

Observe that

$$\frac{x^2}{t_2-t_1}+\frac{(y-x)^2}{t_3-t_2}+\frac{y^2}{t_4-t_3} = \frac{t_3-t_1}{(t_2-t_1)(t_3-t_2)}\left(x-\frac{t_2-t_1}{t_3-t_1}\,y\right)^2+\frac{t_4-t_1}{(t_3-t_1)(t_4-t_3)}\,y^2.$$

Therefore, we get (using physicists' notation: $\int dy\,h(y) := \int h(y)\,dy$ for easier readability)

$$\begin{aligned}
&\iint xy\, f_{B(t_2),B(t_3)|B(t_1),B(t_4)}(x,y\,|\,B(t_1)=B(t_4)=0)\,dx\,dy\\
&= \frac{1}{2\pi(t_4-t_3)}\int_{y=-\infty}^{\infty} dy\exp\left[-\frac12\,\frac{t_4-t_1}{(t_3-t_1)(t_4-t_3)}\,y^2\right]\times\\
&\quad\times \underbrace{\frac{y}{\sqrt{2\pi(t_2-t_1)(t_3-t_2)}}\int_{x=-\infty}^{\infty} x\,\exp\left[-\frac12\left(x-\frac{t_2-t_1}{t_3-t_1}\,y\right)^2\frac{t_3-t_1}{(t_2-t_1)(t_3-t_2)}\right]dx}_{=\frac{y^2}{\sqrt{t_3-t_1}}\,\frac{t_2-t_1}{t_3-t_1}}\\
&= \frac{t_2-t_1}{t_3-t_1}\,\frac{(t_4-t_3)(t_3-t_1)}{t_4-t_1} = \frac{(t_2-t_1)(t_4-t_3)}{t_4-t_1}.
\end{aligned}$$

Problem 2.7 (Solution) Let $s \leqslant t$. Then

$$\begin{aligned}
C(s,t) &= \mathbb{E}(X_sX_t)\\
&= \mathbb{E}(B_s^2-s)(B_t^2-t)\\
&= \mathbb{E}(B_s^2-s)([B_t-B_s+B_s]^2-t)\\
&= \mathbb{E}(B_s^2-s)(B_t-B_s)^2+2\,\mathbb{E}(B_s^2-s)B_s(B_t-B_s)+\mathbb{E}(B_s^2-s)B_s^2-\mathbb{E}(B_s^2-s)t\\
&\overset{\text{(B1)}}{=} \mathbb{E}(B_s^2-s)\,\mathbb{E}(B_t-B_s)^2+2\,\mathbb{E}(B_s^2-s)B_s\,\mathbb{E}(B_t-B_s)+\mathbb{E}(B_s^2-s)B_s^2-\mathbb{E}(B_s^2-s)t\\
&= 0\cdot(t-s)+2\,\mathbb{E}(B_s^2-s)B_s\cdot 0+\mathbb{E}\,B_s^4-s\,\mathbb{E}\,B_s^2-0\\
&= 2s^2 = 2(s^2\wedge t^2) = 2(s\wedge t)^2.
\end{aligned}$$

Problem 2.8 (Solution) a) We have for $s,t \geqslant 0$

$$\begin{aligned}
m(t) &= \mathbb{E}\,X_t = e^{-\alpha t/2}\,\mathbb{E}\,B_{e^{\alpha t}} = 0.\\
C(s,t) &= \mathbb{E}(X_sX_t) = e^{-\frac{\alpha}{2}(s+t)}\,\mathbb{E}\,B_{e^{\alpha s}}B_{e^{\alpha t}} = e^{-\frac{\alpha}{2}(s+t)}\left(e^{\alpha s}\wedge e^{\alpha t}\right) = e^{-\frac{\alpha}{2}|t-s|}.
\end{aligned}$$

b) We have

$$\mathbb{P}(X(t_1)\leqslant x_1,\dots,X(t_n)\leqslant x_n) = \mathbb{P}\left(B(e^{\alpha t_1})\leqslant e^{\alpha t_1/2}x_1,\dots,B(e^{\alpha t_n})\leqslant e^{\alpha t_n/2}x_n\right)$$

Thus, the density is

$$\begin{aligned}
&f_{X(t_1),\dots,X(t_n)}(x_1,\dots,x_n)\\
&= \prod_{k=1}^n e^{\alpha t_k/2} f_{B(e^{\alpha t_1}),\dots,B(e^{\alpha t_n})}(e^{\alpha t_1/2}x_1,\dots,e^{\alpha t_n/2}x_n)
\end{aligned}$$

$$= \prod_{k=1}^{n} e^{\alpha t_k/2}(2\pi)^{-n/2}\left(\prod_{k=1}^{n}(e^{\alpha t_k}-e^{\alpha t_{k-1}})\right)^{-1/2} e^{-\frac{1}{2}\sum_{k=1}^{n}(e^{\alpha t_k/2}x_k - e^{\alpha t_{k-1}/2}x_{k-1})^2/(e^{\alpha t_k}-e^{\alpha t_{k-1}})}$$

$$= (2\pi)^{-n/2}\left(\prod_{k=1}^{n}(1-e^{-\alpha(t_k-t_{k-1})})\right)^{-1/2} e^{-\frac{1}{2}\sum_{k=1}^{n}(x_k - e^{-\alpha(t_k-t_{k-1})/2}x_{k-1})^2/(1-e^{\alpha(t_k-t_{k-1})})}$$

(we use the convention $t_0 = -\infty$ and $x_0 = 0$).

Remark: the form of the density shows that the Ornstein–Uhlenbeck is strictly stationary, i.e.

$$(X(t_1+h),\dots,X(t_n+h)) \sim (X(t_1),\dots,X(t_n)) \qquad \forall h > 0.$$

Problem 2.9 (Solution) "$\Rightarrow$" Assume that we have (B1). Observe that the family of sets

$$\bigcup_{0\leqslant u_1\leqslant\dots\leqslant u_n\leqslant s,\ n\geqslant 1} \sigma(B_{u_1},\dots,B_{u_n})$$

is a $\cap$-stable family. This means that it is enough to show that

$$B_t - B_s \perp\!\!\!\perp (B_{u_1},\dots,B_{u_n}) \quad \text{for all} \quad t \geqslant s \geqslant 0.$$

By (B1) we know that

$$B_t - B_s \perp\!\!\!\perp (B_{u_1}, B_{u_2}-B_{u_1},\dots,B_{u_n}-B_{u_{n-1}})$$

and so

$$B_t - B_s \perp\!\!\!\perp \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix} \begin{pmatrix} B_{u_1} \\ B_{u_2}-B_{u_1} \\ B_{u_3}-B_{u_2} \\ \vdots \\ B_{u_n}-B_{u_{n-1}} \end{pmatrix} = \begin{pmatrix} B_{u_1} \\ B_{u_2} \\ B_{u_3} \\ \vdots \\ B_{u_n} \end{pmatrix}$$

"$\Leftarrow$" Let $0 = t_0 \leqslant t_1 < t_2 < \ldots < t_n < \infty$, $n \geqslant 1$. Then we find for all $\xi_1,\dots,\xi_n \in \mathbb{R}^d$

$$\begin{aligned}
\mathbb{E}\left(e^{i\sum_{k=1}^{n}\langle \xi_k,\, B(t_k)-B(t_{k-1})\rangle}\right) &= \mathbb{E}\Big(e^{i\langle \xi_n,\, B(t_n)-B(t_{n-1})\rangle}\cdot \underbrace{e^{i\sum_{k=1}^{n-1}\langle \xi_k,\, B(t_k)-B(t_{k-1})\rangle}}_{\mathcal{F}_{t_{n-1}}\text{ mble., hence }\perp\!\!\!\perp B(t_n)-B(t_{n-1})}\Big) \\
&= \mathbb{E}\left(e^{i\langle \xi_n,\, B(t_n)-B(t_{n-1})\rangle}\right)\cdot \mathbb{E}\left(e^{i\sum_{k=1}^{n-1}\langle \xi_k,\, B(t_k)-B(t_{k-1})\rangle}\right) \\
&\ \ \vdots \\
&= \prod_{k=1}^{n}\mathbb{E}\left(e^{i\langle \xi_k,\, B(t_k)-B(t_{k-1})\rangle}\right).
\end{aligned}$$

This shows (B1).

Problem 2.10 (Solution) Reflection invariance of BM, cf. 2.8, shows

$$\tau_a = \inf\{s \geqslant 0 : B_s = a\} \sim \inf\{s \geqslant 0 : -B_s = a\} = \inf\{s \geqslant 0 : B_s = -a\} = \tau_{-a}.$$

The scaling property 2.12 of BM shows for $c = 1/a^2$

$$\begin{aligned}
\tau_a = \inf\{s \geqslant 0 : B_s = a\} &\sim \inf\{s \geqslant 0 : aB_{s/a^2} = a\} \\
&= \inf\{a^2 r \geqslant 0 : aB_r = a\} \\
&= a^2 \inf\{r \geqslant 0 : B_r = 1\} = a^2\tau_1.
\end{aligned}$$

Problem 2.11 (Solution) a) **Not stationary:**

$$\mathbb{E}\, W_t^2 = C(t,t) = \mathbb{E}(B_t^2 - t)^2 = \mathbb{E}(B_t^4 - 2tB_t^2 + t^2) = 3t^2 - 2t^2 + t^2 = 2t^2 \neq \text{const}.$$

b) **Stationary.** We have $\mathbb{E}\, X_t = 0$ and

$$\mathbb{E}\, X_s X_t = e^{-\alpha(t+s)/2}\, \mathbb{E}\, B_{e^{\alpha s}} B_{e^{\alpha t}} = e^{-\alpha(t+s)/2}\big(e^{\alpha s} \wedge e^{\alpha t}\big) = e^{-\alpha|t-s|/2},$$

i.e. it is stationary with $g(r) = e^{-\alpha|r|/2}$.

c) **Stationary.** We have $\mathbb{E}\, Y_t = 0$. Let $s \leqslant t$. Then we use $\mathbb{E}\, B_s B_t = s \wedge t$ to get

$$\begin{aligned}
\mathbb{E}\, Y_s Y_t &= \mathbb{E}(B_{s+h} - B_s)(B_{t+h} - B_t) \\
&= \mathbb{E}\, B_{s+h}B_{t+h} - \mathbb{E}\, B_{s+h}B_t - \mathbb{E}\, B_s B_{t+h} + \mathbb{E}\, B_s B_t \\
&= (s+h) \wedge (t+h) - (s+h) \wedge t - s \wedge (t+h) + s \wedge t \\
&= (s+h) - (s+h) \wedge t = \begin{cases} 0, & \text{if } t > s+h \iff h < t-s \\ h - (t-s), & \text{if } t \leqslant s+h \iff h \geqslant t-s. \end{cases}
\end{aligned}$$

Swapping the roles of s and t finally gives: the process is stationary with $g(t) = (h - |t|)^+ = (h - |t|) \vee 0$.

d) **Not stationary.** Note that

$$\mathbb{E}\, Z_t^2 = \mathbb{E}\, B_{e^t}^2 = e^t \neq \text{const}.$$

Problem 2.12 (Solution) Clearly, $t \mapsto W_t$ is continuous for $t \neq 1$. If $t = 1$ we get

$$\lim_{t \uparrow 1} W_t(\omega) = W_1(\omega) = B_1(\omega)$$

and

$$\lim_{t \downarrow 1} W_t(\omega) = B_1(\omega) - \lim_{t \downarrow 1} t\beta_{1/t}(\omega) - \beta_1(\omega) = B_1(\omega);$$

this proves continuity for $t = 1$.

Let us check that W is a Gaussian process with $\mathbb{E}\, W_t = 0$ and $\mathbb{E}\, W_s W_t = s \wedge t$. By Corollary 2.7, W is a BM^1.

Pick $n \geqslant 1$ and $t_0 = 0 < t_1 < \ldots < t_n$.

$\boxed{\text{Case 1:}}$ If $t_n \leqslant 1$, there is nothing to show since $(B_t)_{t \in [0,1]}$ is a BM^1.

$\boxed{\text{Case 2:}}$ Assume that $t_n > 1$. Then we have

$$\begin{pmatrix} W_{t_1} \\ W_{t_2} \\ \vdots \\ W_{t_n} \end{pmatrix} = \begin{pmatrix} 1 & t_1 & 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & t_2 & 0 & \cdots & 0 & -1 \\ \vdots & \vdots & 0 & t_3 & \cdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & t_n & -1 \end{pmatrix} \begin{pmatrix} B_1 \\ \beta_{1/t_1} \\ \vdots \\ \beta_{1/t_n} \\ \beta_1 \end{pmatrix}$$

and since

$$B_1 \perp\!\!\!\perp (\beta_{1/t_1}, \dots, \beta_{1/t_n}, \beta_1)^\top$$

are both Gaussian, we see that $(W_{t_1}, \dots, W_{t_n})$ is Gaussian.

Further, let $t \geqslant 1$ and $1 \leqslant t_i < t_j$:

$$\begin{aligned}
\mathbb{E}\, W_t &= \mathbb{E}\, B_1 + t\,\mathbb{E}\,\beta_{1/t} - \mathbb{E}\,\beta_1 = 0\\
\mathbb{E}\, W_{t_i} W_{t_j} &= \mathbb{E}(B_1 + t_i\beta_{1/t_i} - \beta_1)(B_1 + t_j\beta_{1/t_j} - \beta_1)\\
&= 1 + t_i t_j t_j^{-1} - t_i t_i^{-1} - t_j t_j^{-1} + 1 = t_i = t_i \wedge t_j.
\end{aligned}$$

$\boxed{\text{Case 3:}}$ Assume that $0 < t_1 < \dots < t_k \leqslant 1 < t_{k+1} < \dots < t_n$. Then we have

$$\begin{pmatrix} W_{t_1}\\ W_{t_2}\\ \vdots\\ W_{t_k}\\ \vdots\\ W_{t_n}\end{pmatrix} = \begin{pmatrix}
1 & 0 & \cdots & 0 & & & & & & \\
0 & \ddots & & 0 & & & & & & \\
\vdots & & \ddots & \vdots & & & & & & \\
0 & 0 & \cdots & 1 & & & & & & \\
 & & & & 1 & t_{k+1} & 0 & \cdots & 0 & -1\\
 & & & & 1 & 0 & t_{k+2} & & 0 & -1\\
 & & & & \vdots & \vdots & & \ddots & & \vdots\\
 & & & & 1 & 0 & \cdots & & t_n & -1
\end{pmatrix}\begin{pmatrix} B_{t_1}\\ \vdots\\ \vdots\\ B_{t_k}\\ B_1\\ \beta_{1/t_{k+1}}\\ \vdots\\ \beta_{1/t_n}\\ \beta_1\end{pmatrix}.$$

Since

$$(B_{t_1}, \dots, B_{t_k}, B_1) \perp\!\!\!\perp (\beta_{1/t_{k+1}}, \dots, \beta_{1/t_n}, \beta_1)$$

are Gaussian vectors, $(W_{t_1}, \dots, W_{t_n})$ is also Gaussian and we find

$$\begin{aligned}
\mathbb{E}\, W_t &= 0\\
\mathbb{E}\, W_{t_i} W_{t_j} &= \mathbb{E}\, B_{t_i}(B_1 + t_j\beta_{1/t_j} - \beta_1) = t_i = t_i \wedge t_j
\end{aligned}$$

for $i \leqslant k < j$.

Problem 2.13 (Solution) The process $X(t) = B(e^t)$ has no memory since (cf. Problem 2.9)

$$\sigma(B(s) : s \leqslant e^a) \perp\!\!\!\perp \sigma(B(s) - B(e^a) : s \geqslant e^a)$$

and, therefore,

$$\begin{aligned}
\sigma(X(t) : t \leqslant a) = \sigma(B(s) : 1 \leqslant s \leqslant e^a) &\perp\!\!\!\perp \sigma(B(e^{a+s}) - B(e^a) : s \geqslant 0)\\
&= \sigma(X(t+a) - X(a) : t \geqslant 0).
\end{aligned}$$

The process $X(t) := e^{-t/2}B(e^t)$ is not memoryless. For example, $X(a+a) - X(a)$ is not independent of $X(a)$:

$$\mathbb{E}(X(2a) - X(a))X(a) = \mathbb{E}\left(e^{-a}B(e^{2a}) - e^{-a/2}B(e^a)\right)e^{-a/2}B(e^a) = e^{-3a/2}e^a - e^{-a}e^a \neq 0.$$

Problem 2.14 (Solution) The process $W_t = B_{a-t} - B_a, 0 \leqslant t \leqslant a$ clearly satisfies (B0) and (B4).

For $0 \leqslant s \leqslant t \leqslant a$ we find

$$W_t - W_s = B_{a-t} - B_{a-s} \sim B_{a-s} - B_{a-t} \sim B_{t-s} \sim \mathsf{N}(0, (t-s)\,\mathrm{id})$$

and this shows (B2) and (B3).

For $0 = t_0 < t_1 < \ldots < t_n \leqslant a$ we have

$$W_{t_j} - W_{t_{j-1}} = B_{a-t_j} - B_{a-t_{j-1}} \sim B_{a-t_{j-1}} - B_{a-t_j} \quad \forall j$$

and this proves that W inherits (B1) from B.

Problem 2.15 (Solution) We know from Paragraph 2.13 that

$$\lim_{t\downarrow 0} tB(1/t) = 0 \implies \lim_{s\uparrow\infty} \frac{B(s)}{s} = 0 \quad \text{a.s.}$$

Moreover,

$$\mathbb{E}\left(\frac{B(s)}{s}\right)^2 = \frac{s}{s^2} = \frac{1}{s} \xrightarrow{s\to\infty} 0$$

i.e. we get also convergence in mean square.

Remark: a direct proof of the SLLN is a bit more tricky. Of course we have by the classical SLLN that

$$\frac{B_n}{n} = \frac{\sum_{j=1}^n (B_j - B_{j-1})}{n} \xrightarrow[n\to\infty]{\text{SLLN}} 0 \quad \text{a.s.}$$

But then we have to make sure that B_s/s converges. This can be done in the following way: fix $s > 0$. Then there is a unique interval $(n, n+1]$ such that $s \in (n, n+1]$. Thus,

$$\left|\frac{B_s}{s}\right| \leqslant \left|\frac{B_s - B_{n+1}}{s}\right| + \left|\frac{B_{n+1}}{n+1}\right| \cdot \frac{n+1}{s} \leqslant \frac{\sup_{n\leqslant s\leqslant n+1}|B_s - B_{n+1}|}{n} + \frac{n+1}{n}\left|\frac{B_n}{n}\right|$$

and we have to show that the expression with the sup tends to zero. This can be done by showing, e.g., that the L^2-limit of this expression goes to zero (using the reflection principle) and with a subsequence argument.

Problem 2.16 (Solution) Set

$$\Sigma := \bigcup_{J\subset[0,\infty),\ J \text{ countable}} \sigma(B(t) : t \in J)$$

Clearly,

$$\bigcup_{t\geqslant 0} \sigma(B_t) \subset \Sigma \subset \sigma(B_t : t \geqslant 0) \overset{\text{def}}{=} \mathcal{F}_\infty^B \tag{*}$$

The first inclusion follows from the fact that each B_t is measurable with respect to Σ.

Let us show that Σ is a σ-algebra. Obviously,

$$\varnothing \in \Sigma \quad \text{and} \quad F \in \Sigma \implies F^c \in \Sigma.$$

Let $(A_n)_n \subset \Sigma$. Then, for every n there is a countable set J_n such that $A_n \in \sigma(B(t) : t \in J_n)$. Since $J = \bigcup_n J_n$ is still countable we see that $A_n \in \sigma(B(t) : t \in J)$ for all n. Since the latter family is a σ-algebra, we find

$$\bigcup_n A_n \in \sigma(B(t) : t \in J) \subset \Sigma.$$

Since $\bigcup_t \sigma(B_t) \subset \Sigma$, we get—note: $\mathcal{F}_\infty^B$ is, by definition, the smallest σ-algebra for which all B_t are measurable—that

$$\mathcal{F}_\infty^B \subset \Sigma.$$

This shows that $\Sigma = \mathcal{F}_\infty^B$.

Problem 2.17 (Solution) Assume that the indices $t_1, \dots, t_m$ and $s_1, \dots, s_n$ are given. Let $\{u_1, \dots, u_p\} := \{s_1, \dots, s_n\} \cup \{t_1, \dots, t_m\}$. By assumption,

$$(X(u_1), \dots, X(u_p)) \perp\!\!\!\perp (Y(u_1), \dots, Y(u_p)).$$

Thus, we may thin out the indices on each side without endangering independence: $\{s_1, \dots, s_n\} \subset \{u_1, \dots, u_p\}$ and $\{t_1, \dots, t_m\} \subset \{u_1, \dots, u_p\}$, and so

$$(X(s_1), \dots, X(s_n)) \perp\!\!\!\perp (Y(t_1), \dots, Y(t_m)).$$

Problem 2.18 (Solution) Since $\mathcal{F}_t \subset \mathcal{F}_\infty$ and $\mathcal{G}_t \subset \mathcal{G}_\infty$ it is clear that

$$\mathcal{F}_\infty \perp\!\!\!\perp \mathcal{G}_\infty \implies \mathcal{F}_t \perp\!\!\!\perp \mathcal{G}_t.$$

Conversely, since $(\mathcal{F}_t)_{t\geqslant 0}$ and $(\mathcal{G}_t)_{t\geqslant 0}$ are filtrations we find

$$\forall F \in \bigcup_{t\geqslant 0} \mathcal{F}_t, \quad \forall G \in \bigcup_{t\geqslant 0} \mathcal{G}_t, \quad \exists t_0 : F \in \mathcal{F}_{t_0},\, G \in \mathcal{G}_{t_0}.$$

By assumption: $\mathbb{P}(F \cap G) = \mathbb{P}(F)\,\mathbb{P}(G)$. Thus,

$$\bigcup_{t\geqslant 0} \mathcal{F}_t \perp\!\!\!\perp \bigcup_{t\geqslant 0} \mathcal{G}_t.$$

Since the families $\bigcup_{t\geqslant 0} \mathcal{F}_t$ and $\bigcup_{t\geqslant 0} \mathcal{G}_t$ are $\cap$-stable (use again the argument that we have filtrations to find for $F, F' \in \bigcup_{t\geqslant 0} \mathcal{F}_t$ some t_0 with $F, F' \in \mathcal{F}_{t_0}$ etc.), the σ-algebras generated by these families are independent:

$$\mathcal{F}_\infty = \sigma\left(\bigcup_{t\geqslant 0} \mathcal{F}_t\right) \perp\!\!\!\perp \sigma\left(\bigcup_{t\geqslant 0} \mathcal{G}_t\right) = \mathcal{G}_\infty.$$

Problem 2.19 (Solution) Let $U \in \mathbb{R}^{d\times d}$ be an orthogonal matrix: $UU^\top = \mathrm{id}$ and set $X_t := UB_t$ for a BM^d $(B_t)_{t\geqslant 0}$. Then

$$\mathbb{E}\left(\exp\left[i\sum_{j=1}^n \langle \xi_j,\, X(t_j) - X(t_{j-1})\rangle\right]\right) = \mathbb{E}\left(\exp\left[i\sum_{j=1}^n \langle \xi_j,\, UB(t_j) - UB(t_{j-1})\rangle\right]\right)$$

$$= \mathbb{E}\left(\exp\left[i\sum_{j=1}^{n}\langle U^\top\xi_j,\, B(t_j)-B(t_{j-1})\rangle\right]\right)$$
$$= \exp\left[-\frac{1}{2}\sum_{j=1}^{n}(t_j-t_{j-1})\langle U^\top\xi_j,\, U^\top\xi_j\rangle\right]$$
$$= \exp\left[-\frac{1}{2}\sum_{j=1}^{n}(t_j-t_{j-1})|\xi_j|^2\right].$$

(Observe $\langle U^\top\xi_j, U^\top\xi_j\rangle = \langle UU^\top\xi_j, \xi_j\rangle = \langle\xi_j, \xi_j\rangle = |\xi_j|^2$). The claim follows.

Problem 2.20 (Solution) Note that the coordinate processes b and β are independent BM^1.

a) Since $b \perp\!\!\!\perp \beta$, the process $W_t = (b_t + \beta_t)/\sqrt{2}$ is a Gaussian process with continuous sample paths. We determine its mean and covariance functions:

$$\begin{aligned}
\mathbb{E}\,W_t &= \frac{1}{\sqrt{2}}(\mathbb{E}\,b_t + \mathbb{E}\,\beta_t) = 0;\\
\operatorname{Cov}(W_s, W_t) &= \mathbb{E}(W_sW_t)\\
&= \frac{1}{2}\,\mathbb{E}(b_s+\beta_s)(b_t+\beta_t)\\
&= \frac{1}{2}\Big(\mathbb{E}\,b_sb_t + \mathbb{E}\,\beta_sb_t + \mathbb{E}\,b_s\beta_t + \mathbb{E}\,\beta_s\beta_t\Big)\\
&= \frac{1}{2}\Big(s\wedge t + 0 + 0 + s\wedge t\Big) = s\wedge t
\end{aligned}$$

where we used that, by independence, $\mathbb{E}\,b_u\beta_v = \mathbb{E}\,b_u\,\mathbb{E}\,\beta_v = 0$. Now the claim follows from Corollary 2.7.

b) The process $X_t = (W_t, \beta_t)$ has the following properties

- W and β are BM^1
- $\mathbb{E}(W_tb_t) = 2^{-1/2}\,\mathbb{E}(b_t+\beta_t)\beta_t = 2^{-1/2}(\mathbb{E}\,b_t\,\mathbb{E}\,\beta_t + \mathbb{E}\,\beta_t^2) = t/\sqrt{2} \neq 0$, i.e. W and β are NOT independent.

This means that X is not a BM^2, as its coordinates are not independent.

The process Y_t can be written as

$$\frac{1}{\sqrt{2}}\begin{pmatrix} b_t+\beta_t\\ b_t-\beta_t\end{pmatrix} = U\begin{pmatrix} b_t\\ \beta_t\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix} b_t\\ \beta_t\end{pmatrix}.$$

Clearly, $UU^\top = \mathrm{id}$, i.e. Problem 2.19 shows that $(Y_t)_{t\geqslant 0}$ is a BM^2.

Problem 2.21 (Solution) Observe that $b \perp\!\!\!\perp \beta$ since B is a BM^2. Since

$$\begin{aligned}
\mathbb{E}\,X_t &= 0\\
\operatorname{Cov}(X_t, X_s) &= \mathbb{E}\,X_tX_s\\
&= \mathbb{E}(\lambda b_s + \mu\beta_s)(\lambda b_t + \mu\beta_t)\\
&= \lambda^2\,\mathbb{E}\,b_sb_t + \lambda\mu\,\mathbb{E}\,b_s\beta_t + \lambda\mu\,\mathbb{E}\,b_t\beta_s + \mu^2\beta_s\beta_t
\end{aligned}$$

$$= \lambda^2 \,\mathbb{E}\, b_s b_t + \lambda\mu \,\mathbb{E}\, b_s \,\mathbb{E}\, \beta_t + \lambda\mu \,\mathbb{E}\, b_t \,\mathbb{E}\, \beta_s + \mu^2 \,\mathbb{E}\, \beta_s \beta_t$$
$$= \lambda^2 (s \wedge t) + 0 + 0 + \mu^2 s \wedge t = (\lambda^2 + \mu^2)(s \wedge t).$$

Thus, by Corollary 2.7, X is a BM^1 if, and only if, $\lambda^2 + \mu^2 = 1$.

Problem 2.22 (Solution) $X_t = (b_t, \beta_{s-t} - \beta_t)$, $0 \leqslant t \leqslant s$, is NOT a Brownian motion: $X_0 = (0, \beta_s) \neq (0,0)$.

On the other hand, $Y_t = (b_t, \beta_{s-t} - \beta_s)$, $0 \leqslant t \leqslant s$, IS a Brownian motion, since b_t and $\beta_{s-t} - \beta_s$ are independent BM^1, cf. Time inversion 2.11 and Theorem 2.16.

Problem 2.23 (Solution) We have

$$W_t = U B_t^\top = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} b_t \\ \beta_t \end{pmatrix}.$$

The matrix U is a rotation, hence orthogonal and we see from Problem 2.19 that W is a Brownian motion.

Generalization: take U orthogonal.

Problem 2.24 (Solution) If $G \sim \mathsf{N}(0,Q)$ then Q is the covariance matrix, i.e. $\operatorname{Cov}(G^j, G^k) = q_{jk}$. Thus, we get for $s < t$

$$\begin{aligned} \operatorname{Cov}(X_s^j, X_t^k) &= \mathbb{E}(X_s^j X_t^k) \\ &= \mathbb{E}\, X_s^j (X_t^k - X_s^k) + \mathbb{E}(X_s^j X_s^k) \\ &= \mathbb{E}\, X_s^j \,\mathbb{E}(X_t^k - X_s^k) + s q_{jk} \\ &= (s \wedge t) q_{jk}. \end{aligned}$$

The characteristic function is

$$\mathbb{E}\, e^{i\langle \xi, X_t\rangle} = \mathbb{E}\, e^{i\langle \Sigma^\top \xi, B_t\rangle} = e^{-\frac{t}{2}|\Sigma^\top \xi|^2} = e^{-\frac{t}{2}\langle \xi, \Sigma\Sigma^\top \xi\rangle},$$

and the transition probability is, if Q is non-degenerate,

$$f_Q(x) = \frac{1}{\sqrt{(2\pi t)^n \det Q}} \exp\left(-\frac{1}{2t}\langle x, Qx\rangle\right).$$

If Q is degenerate, there is an orthogonal matrix $U \in \mathbb{R}^{n\times n}$ such that

$$U X_t = (Y_t^1, \dots, Y_t^k, \underbrace{0, \dots, 0}_{n-k})^\top$$

where $k < n$ is the rank of Q. The k-dimensional vector has a nondegenerate normal distribution in $\mathbb{R}^k$.

3 Constructions of Brownian Motion

Problem 3.1 (Solution) The partial sums

$$W_N(t,\omega) = \sum_{n=0}^{N-1} G_n(\omega) S_n(t), \quad t \in [0,1],$$

converge as $N \to \infty$ $\mathbb{P}$-a.s. uniformly for t towards $B(t,\omega), t \in [0,1]$—cf. Problem 3.2. Therefore, the random variables

$$\int_0^1 W_N(t)\,dt = \sum_{n=0}^{N-1} G_n \int_0^1 S_n(t)\,dt \xrightarrow[N\to\infty]{\mathbb{P}\text{-a.s.}} X = \int_0^1 B(t)\,dt.$$

This shows that $\int_0^1 W_N(t)\,dt$ is the sum of independent $\mathsf{N}(0,1)$-random variables, hence itself normal and so is its limit X.

From the definition of the Schauder functions (cf. Figure 3.2) we find

$$\int_0^1 S_0(t)\,dt = \frac{1}{2}$$
$$\int_0^1 S_{2^j+k}(t)\,dt = \frac{1}{4}\,2^{-\frac{3}{2}j}, \quad k = 0,1,\dots,2^j-1,\ j \geqslant 0.$$

and this shows

$$\int_0^1 W_{2^{n+1}}(t)\,dt = \frac{1}{2}\,G_0 + \frac{1}{4}\sum_{j=0}^{n}\sum_{l=0}^{2^j-1} 2^{-\frac{3}{2}j} G_{2^j+l}.$$

Consequently, since the G_j are iid $\mathsf{N}(0,1)$ random variables,

$$\mathbb{E}\int_0^1 W_{2^{n+1}}(t)\,dt = 0,$$

$$\begin{aligned}
\mathbb{V}\int_0^1 W_{2^{n+1}}(t)\,dt &= \frac{1}{4} + \frac{1}{16}\sum_{j=0}^{n}\sum_{l=0}^{2^j-1} 2^{-3j} \\
&= \frac{1}{4} + \frac{1}{16}\sum_{j=0}^{n} 2^{-2j} \\
&= \frac{1}{4} + \frac{1}{16}\,\frac{1-2^{-2(n+1)}}{1-\frac{1}{4}} \\
&\xrightarrow[n\to\infty]{} \frac{1}{4} + \frac{1}{16}\,\frac{4}{3} = \frac{1}{3}.
\end{aligned}$$

This means that

$$X = \frac{1}{2}\,G_0 + \sum_{j=0}^{\infty}\frac{1}{4}\,2^{-\frac{3}{2}j}\underbrace{\sum_{l=0}^{2^j-1} G_{2^j+l}}_{\sim\mathsf{N}(0,2^j)}$$

where the series converges $\mathbb{P}$-a.s. and in mean square, and $X \sim \mathsf{N}(0,\frac{1}{3})$.

Problem 3.2 (Solution) a) From the definition of the Schauder functions $S_n(t)$, $n \geqslant 0$, $t \in [0,1]$, we find

$$\begin{aligned} 0 \leqslant S_n(t) &\qquad \forall n,\, t \\ S_{2^j+k}(t) \leqslant S_{2^j+k}\big((2k+1)/2^{j+1}\big) = 2^{-j/2}/2^{j+1} = \frac{1}{2}\,2^{-j/2} &\qquad \forall j,k,t \\ \sum_{k=0}^{2^j-1} S_{2^j+k}(t) \leqslant \frac{1}{2}\,2^{-j/2} &\qquad \text{(disjoint supports!)} \end{aligned}$$

By assumption,

$$\exists C>0, \quad \exists \epsilon \in (0,\tfrac{1}{2}), \quad \forall n : |a_n| \leqslant C \cdot n^{\epsilon}.$$

Thus, we find

$$\begin{aligned} \sum_{n=0}^{\infty} |a_n| S_n(t) &\leqslant |a_0| + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} |a_{2^j+k}| S_{2^j+k}(t) \\ &\leqslant |a_0| + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} C \cdot (2^{j+1})^{\epsilon}\, S_{2^j+k}(t) \\ &\leqslant |a_0| + \sum_{j=0}^{\infty} C \cdot 2^{(j+1)\epsilon}\, \frac{1}{2}\, 2^{-j} < \infty. \end{aligned}$$

(The series is convergent since $\epsilon < 1/2$).

This shows that $\sum_{n=0}^{\infty} a_n S_n(t)$ converges absolutely and uniformly for $t \in [0,1]$.

b) For $C > \sqrt{2}$ we find from

$$\mathbb{P}\left(|G_n| > \sqrt{\log n}\right) \leqslant \sqrt{\frac{2}{\pi}}\, \frac{1}{C\sqrt{\log n}}\, e^{-\frac{1}{2}C^2 \log n} \leqslant \sqrt{\frac{2}{\pi}}\, \frac{1}{C}\, n^{-C^2/2} \qquad \forall n \geqslant 3$$

that the following series converges:

$$\sum_{n=1}^{\infty} \mathbb{P}\left(|G_n| > \sqrt{\log n}\right) < \infty.$$

By the Borel–Cantelli Lemma we find that $G_n(\omega) = \mathsf{O}\big(\sqrt{\log n}\big)$ for almost all ω, thus $G_n(\omega) = \mathsf{O}\big(n^{\epsilon}\big)$ for any $\epsilon \in (0,1/2)$.

From part a) we know that the series $\sum_{n=0}^{\infty} G_n(\omega) S_n(t)$ converges a.s. uniformly for $t \in [0,1]$.

Problem 3.3 (Solution) Set $\|f\|_p := \big(\mathbb{E}|f|^p\big)^{1/p}$

Solution 1: We observe that the space $L^p(\Omega,\mathcal{A},\mathbb{P};S) = \{X : X \in S, \|d(X,0)\|_p < \infty\}$ is complete and that the condition stated in the problem just says that $(X_n)_n$ is a Cauchy sequence in the space $L^p(\Omega,\mathcal{A},\mathbb{P};S)$. A good reference for this is, for example, the monograph by F. Trèves [13, Chapter 46]. You will find the 'pedestrian' approach as Solution 2 below.

<u>Solution 2:</u> By assumption

$$\forall k \geqslant 0 \quad \exists N_k \geqslant 1 : \sup_{m \geqslant N_k} \|d(X_{N_k}, X_m)\|_p \leqslant 2^{-k}.$$

Without loss of generality we can assume that $N_k \leqslant N_{k+1}$. In particular

$$\|d(X_{N_k}, X_{N_{k+1}})\|_p \leqslant 2^{-k} \overset{\forall l > k}{\Longrightarrow} \|d(X_{N_k}, X_{N_l})\|_p \leqslant \sum_{j=k}^{l-1} 2^{-j} \leqslant \frac{2}{2^k}.$$

Fix $m \geqslant 1$. Then we see that

$$\|d(X_{N_k}, X_m) - d(X_{N_l}, X_m)\|_p \leqslant \|d(X_{N_k}, X_{N_l})\|_p \xrightarrow[k,l \to \infty]{} 0.$$

This means that that $(d(X_{N_k}, X_m))_{k \geqslant 0}$ is a Cauchy sequence in $L^p(\mathbb{P}; \mathbb{R})$. By the completeness of the space $L^p(\mathbb{P}; \mathbb{R})$ there is some $f_m \in L^p(\mathbb{P}; \mathbb{R})$ such that

$$d(X_{N_k}, X_m) \xrightarrow[k \to \infty]{\text{in } L^p} f_m$$

and, for a subsequence $(n_k) \subset (N_k)_k$ we find

$$d(X_{n_k}, X_m) \xrightarrow[k \to \infty]{\text{almost surely}} f_m.$$

The subsequence n_k may also depend on m. Since $(n_k(m))_k$ is still a subsequence of (N_k), we still have $d(X_{n_k(m)}, X_{m+1}) \to f_{m+1}$ in L^p, hence we can find a subsequence $(n_k(m+1))_k \subset (n_k(m))_k$ such that $d(X_{n_k(m+1)}, X_{m+1}) \to f_{m+1}$ a.s. Iterating this we see that we can assume that $(n_k)_k$ does not depend on m.

In particular, we have almost surely

$$\forall \epsilon > 0 \quad \exists L = L(\epsilon) \geqslant 1 \quad \forall k \geqslant L : |d(X_{n_k}, X_m) - f_m| \leqslant \epsilon.$$

Moreover,

$$\begin{aligned} \lim_{m \to \infty} \|f_m\|_p = \lim_{m \to \infty} \| \lim_{k \to \infty} d(X_{n_k}, X_m)\|_p &\leqslant \lim_{m \to \infty} \varliminf_{k \to \infty} \|d(X_{n_k}, X_m)\|_p \\ &\leqslant \varliminf_{k \to \infty} \sup_{m \geqslant n_k} \|d(X_{n_k}, X_m)\|_p = 0. \end{aligned}$$

Thus, $f_m \to 0$ in L^p and, for a subsequence m_k we get

$$\forall \epsilon > 0 \quad \exists K = K(\epsilon) \geqslant 1 \quad \forall r \geqslant K : |f_{m_r}| \leqslant \epsilon.$$

Therefore,

$$\begin{aligned} d(X_{n_k}, X_{n_l}) &\leqslant d(X_{n_k}, X_{m_r}) + d(X_{n_k}, X_{m_r}) \\ &\leqslant |d(X_{n_k}, X_{m_r}) - f_{m_r}| + |d(X_{n_k}, X_{m_r}) - f_{m_r}| + 2|f_{m_r}|. \end{aligned}$$

Fix $\epsilon > 0$ and pick $r > K$. Then let $k, l \to \infty$. This gives

$$d(X_{n_k}, X_{n_l}) \leqslant |d(X_{n_k}, X_{m_r}) - f_{m_r}| + |d(X_{n_k}, X_{m_r}) - f_{m_r}| + 2\epsilon \leqslant 4\epsilon \quad \forall k, l \geqslant L(\epsilon).$$

Since S is complete, this proves that $(X_{n_k})_{k\geqslant 0}$ converges to some $X\in S$ almost surely.

<u>Remark:</u> If we replace the condition of the Problem by

$$\lim_{n\to\infty}\mathbb{E}\left(\sup_{m\geqslant n} d^p(X_n,X_m)\right)=0$$

things become MUCH simpler:

This condition says that the sequence $d_n := \sup_{m\geqslant n} d^p(X_n,X_m)$ converges in $L^p(\mathbb{P};\mathbb{R})$ to zero. Hence there is a subsequence $(n_k)_k$ such that

$$\lim_{k\to\infty}\sup_{m\geqslant n_k} d(X_{n_k},X_m)=0$$

almost surely. This shows that $d(X_{n_k},X_{n_l})\to 0$ as $k,l\to\infty$, i.e. we find by the completeness of the space S that $X_{n_k}\to X$.

Problem 3.4 (Solution) Fix $n\geqslant 1$, $0\leqslant t_1\leqslant\ldots\leqslant t_n$ and Borel sets $A_1,\ldots,A_n$. By assumption, we know that

$$\mathbb{P}(X_t=Y_t)=1\quad\forall t\geqslant 0\implies\mathbb{P}(X_{t_j}=Y_{t_j}\ j=1,\ldots,n)=\mathbb{P}\left(\bigcap_{j=1}^n\left\{X_{t_j}=Y_{t_j}\right\}\right)=1.$$

Thus,

$$\begin{aligned}\mathbb{P}\left(\bigcap_{j=1}^n\left\{X_{t_j}\in A_j\right\}\right)&=\mathbb{P}\left(\bigcap_{j=1}^n\left\{X_{t_j}\in A_j\right\}\cap\bigcap_{j=1}^n\left\{X_{t_j}=Y_{t_j}\right\}\right)\\&=\mathbb{P}\left(\bigcap_{j=1}^n\left\{X_{t_j}\in A_j\right\}\cap\left\{X_{t_j}=Y_{t_j}\right\}\right)\\&=\mathbb{P}\left(\bigcap_{j=1}^n\left\{Y_{t_j}\in A_j\right\}\cap\left\{X_{t_j}=Y_{t_j}\right\}\right)\\&=\mathbb{P}\left(\bigcap_{j=1}^n\left\{Y_{t_j}\in A_j\right\}\right).\end{aligned}$$

Problem 3.5 (Solution) <u>indistinguishable $\implies$ modification:</u>

$$\mathbb{P}(X_t=Y_t\ \forall t\geqslant 0)=1\implies\forall t\geqslant 0:\ \mathbb{P}(X_t=Y_t)=1.$$

<u>modification $\implies$ equivalent:</u> see the previous Problem 3.4

Now assume that I is countable or $t\mapsto X_t, t\mapsto Y_t$ are (left- or right-)continuous.

<u>modification $\implies$ indistinquishable:</u> By assumption, $\mathbb{P}(X_t\neq Y_t)=0$ for any $t\in I$. Let $D\subset I$ be any countable dense subset. Then

$$\mathbb{P}\left(\bigcup_{q\in D}\{X_q\neq Y_q\}\right)\leqslant\sum_{q\in D}\mathbb{P}(X_q\neq Y_q)=0$$

which means that $\mathbb{P}(X_q = Y_q \forall q \in D) = 1$. If I is countable, we are done. In the other case we have, by the density of D,

$$\mathbb{P}(X_t = Y_t\ \forall t \in I) = \mathbb{P}\left(\lim_{D\ni q} X_q = \lim_{D\ni q} Y_q\ \forall t \in I\right) \geqslant \mathbb{P}\left(X_q = Y_q\ \forall q \in D\right) = 1.$$

equivalent $\not\Longrightarrow$ modification: To see this let $(B_t)_{t\geqslant 0}$ and $(W_t)_{t\geqslant 0}$ be two **independent** one-dimensional Brownian motions defined on the **same probability space**. Clearly, these processes have the same finite-dimensional distributions, i.e. they are equivalent. On the other hand, for any $t > 0$

$$\mathbb{P}(B_t = W_t) = \int_{-\infty}^{\infty} \mathbb{P}(B_t = y)\ \mathbb{P}(W_t \in dy) = \int_{-\infty}^{\infty} 0\ \mathbb{P}(W_t \in dy) = 0.$$

Problem 3.6 (Solution) Since $(B_q)_{q\in\mathbb{Q}\cap[0,\infty)}$ is uniformly continuous, there exists a unique process $(B_t)_{t\geqslant 0}$ such that $B_t = \lim_{q\to t} B_q$ and $t \mapsto B_t$ is continuous.

We use the characterization from Lemma 2.14. Its proof shows that we can derive (2.17)

$$\mathbb{E}\left[\exp\left(i\sum_{j=1}^{n}\langle\xi_j, X_{q_j} - X_{q_{j-1}}\rangle + i\langle\xi_0, X_{q_0}\rangle\right)\right] = \exp\left(-\frac{1}{2}\sum_{j=1}^{n}|\xi_j|^2(q_j - q_{j-1})\right)$$

on the basis of (B0)–(B3) for $(B_q)_{q\in\mathbb{Q}\cap[0,\infty)}$ and $q_0, \dots, q_n \in \mathbb{Q}\cap[0,\infty)$.

Now set $t_0 = q_0 = 0$ and pick $t_1, \dots, t_n \in \mathbb{R}$ and approximate each t_j by a rational sequence $q_j^{(k)}, k \geqslant 1$. Since (2.17) holds for $q_j^{(k)}, j = 0, \dots, n$ and every $k \geqslant 0$, we can easily perform the limit $k \to \infty$ on both sides (on the left we use dominated convergence!) since B_t is continuous.

This proves (2.17) for $(B_t)_{t\geqslant 0}$, and since $(B_t)_{t\geqslant 0}$ has continuous paths, Lemma 2.14 proves that $(B_t)_{t\geqslant 0}$ is a BM^1.

Problem 3.7 (Solution) The joint density of $(W(t_0), W(t), W(t_1))$ is

$$f_{t_0,t,t_1}(x_0, x, x_1) = \frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{(t_1 - t)(t - t_0)t_0}}\exp\left(-\frac{1}{2}\left[\frac{(x_1 - x)^2}{t_1 - t} + \frac{(x - x_0)^2}{t - t_0} + \frac{x_0^2}{t_0}\right]\right)$$

while the joint density of $(W(t_0), W(t_1))$ is

$$f_{t_0,t_1}(x_0, x_1) = \frac{1}{(2\pi)}\frac{1}{\sqrt{(t_1 - t_0)t_0}}\exp\left(-\frac{1}{2}\left[\frac{(x_1 - x_0)^2}{t_1 - t_0} + \frac{x_0^2}{t_0}\right]\right).$$

The conditional density of $W(t)$ given $(W(t_0), W(t_1))$ is

$$\begin{aligned} &f_{t\mid t_0,t_1}(x \mid x_1, x_2) \\ &= \frac{f_{t_0,t,t_1}(x_0, x, x_1)}{f_{t_0,t_1}(x_0, x_1)} \\ &= \frac{\frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{(t_1-t)(t-t_0)t_0}}\exp\left(-\frac{1}{2}\left[\frac{(x_1-x)^2}{t_1-t} + \frac{(x-x_0)^2}{t-t_0} + \frac{x_0^2}{t_0}\right]\right)}{\frac{1}{(2\pi)}\frac{1}{\sqrt{(t_1-t_0)t_0}}\exp\left(-\frac{1}{2}\left[\frac{(x_1-x_0)^2}{t_1-t_0} + \frac{x_0^2}{t_0}\right]\right)} \end{aligned}$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{(t_1-t_0)}{(t_1-t)(t-t_0)}} \exp\left(-\frac{1}{2}\left[\frac{(x_1-x)^2}{t_1-t} + \frac{(x-x_0)^2}{t-t_0} - \frac{(x_1-x_0)^2}{t_1-t_0}\right]\right)$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{(t_1-t_0)}{(t_1-t)(t-t_0)}} \exp\left(-\frac{1}{2}\left[\frac{(t-t_0)(x_1-x)^2 + (t_1-t)(x-x_0)^2}{(t_1-t)(t-t_0)} - \frac{(x_1-x_0)^2}{t_1-t_0}\right]\right)$$

Now consider the argument in the square brackets $[\cdots]$ of the exp-function

$$\begin{aligned}
&\left[\frac{(t-t_0)(x_1-x)^2 + (t_1-t)(x-x_0)^2}{(t_1-t)(t-t_0)} - \frac{(x_1-x_0)^2}{t_1-t_0}\right] \\
&= \frac{(t_1-t_0)}{(t_1-t)(t-t_0)}\left[\frac{t-t_0}{t_1-t_0}(x_1-x)^2 + \frac{t_1-t}{t_1-t_0}(x-x_0)^2 - \frac{(t_1-t)(t-t_0)}{(t_1-t_0)^2}(x_1-x_0)^2\right] \\
&= \frac{(t_1-t_0)}{(t_1-t)(t-t_0)}\Bigg[\left(\frac{t-t_0}{t_1-t_0} + \frac{t_1-t}{t_1-t_0}\right)x^2 + \left(\frac{t-t_0}{t_1-t_0} - \frac{(t_1-t)(t-t_0)}{(t_1-t_0)^2}\right)x_1^2 \\
&\qquad\qquad + \left(\frac{t_1-t}{t_1-t_0} - \frac{(t_1-t)(t-t_0)}{(t_1-t_0)^2}\right)x_0^2 \\
&\qquad\qquad - 2\frac{t-t_0}{t_1-t_0}x_1x - 2\frac{t_1-t}{t_1-t_0}xx_0 + 2\frac{(t_1-t)(t-t_0)}{(t_1-t_0)^2}x_1x_0\Bigg] \\
&= \frac{(t_1-t_0)}{(t_1-t)(t-t_0)}\Bigg[x^2 + \frac{(t-t_0)^2}{(t_1-t_0)^2}x_1^2 + \frac{(t_1-t)^2}{(t_1-t_0)^2}x_0^2 \\
&\qquad\qquad - 2\frac{t-t_0}{t_1-t_0}x_1x - 2\frac{t_1-t}{t_1-t_0}xx_0 + 2\frac{(t_1-t)(t-t_0)}{(t_1-t_0)^2}x_1x_0\Bigg] \\
&= \frac{(t_1-t_0)}{(t_1-t)(t-t_0)}\left[x - \frac{t-t_0}{t_1-t_0}x_1 - \frac{t_1-t}{t_1-t_0}x_0\right]^2 \\
&= \frac{(t_1-t_0)}{(t_1-t)(t-t_0)}\left[x - \left(\frac{t-t_0}{t_1-t_0}x_1 + \frac{t_1-t}{t_1-t_0}x_0\right)\right]^2.
\end{aligned}$$

Set

$$\sigma^2 = \frac{(t_1-t)(t-t_0)}{(t_1-t_0)} \quad \text{and} \quad m = \frac{t-t_0}{t_1-t_0}x_1 + \frac{t_1-t}{t_1-t_0}x_0$$

then our calculation shows that

$$f_{t\,|\,t_0,t_1}(x\,|\,x_1,x_2) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(\frac{(x-m)^2}{2\sigma^2}\right).$$

4 The Canonical Model

Problem 4.1 (Solution) Let $F : \mathbb{R} \to [0,1]$ be a distribution function. We begin with a general lemma: F has a unique generalized monotone increasing right-continuous inverse:

$$F^{-1}(u) = G(u) = \inf\{x : F(x) > u\} \\ \Big[= \sup\{x : F(x) \leqslant u\} \Big]. \tag{4.1}$$

We have $F(G(u)) = u$ if $F(t)$ is continuous in $t = G(u)$, otherwise, $F(G(u)) \geqslant u$.

Indeed: For those t where F is strictly increasing and continuous, there is nothing to show. Let us look at the two problem cases: F jumps and F is flat.

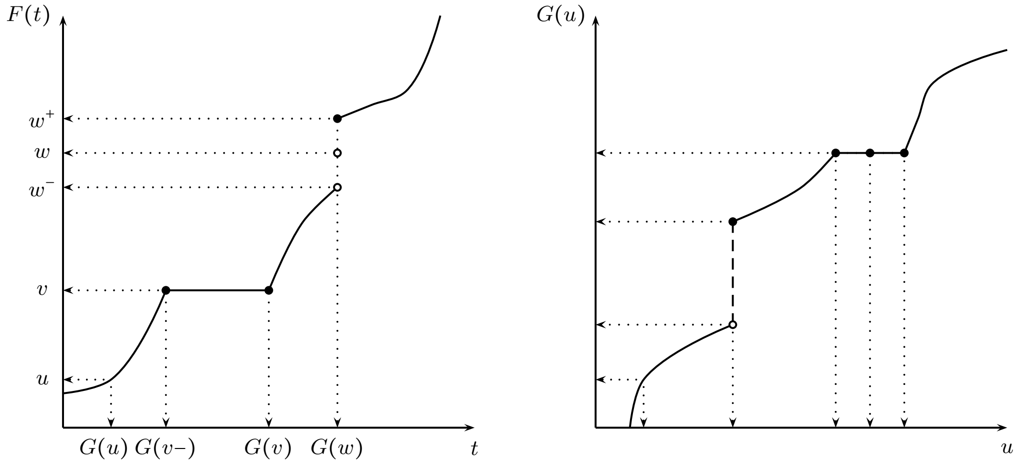

Figure 4.1: An illustration of the problem cases

If $F(t)$ jumps, we have $G(w) = G(w^+) = G(w^-)$ and if $F(t)$ is flat, we take the right endpoint of the 'flatness interval' $[G(v-), G(v)]$ to define G (this leads to right-continuity of G)

a) Let $(\Omega, \mathcal{A}, \mathbb{P}) = ([0,1], \mathcal{B}[0,1], du)$ (du stands for Lebesgue measure) and define $X = G$ ($G = F^{-1}$ as before). Then

$$\begin{aligned} &\mathbb{P}(\{\omega \in \Omega : X(\omega) \leqslant x\}) \\ &\qquad = \lambda(\{u \in [0,1] : G(u) \leqslant x\}) \\ &\qquad\qquad \text{(the discontinuities of } F \text{ are countable, i.e. a Lebesgue null set)} \end{aligned}$$

$$
\begin{aligned}
&= \lambda(\{t \in [0,1] : t \leqslant F(x)\}) \\
&= \lambda([0, F(x)]) = F(x).
\end{aligned}
$$

Measurability is clear because of monotonicity.

b) Use the product construction and part a). To be precise, we do the construction for two random variables. Let $X : \Omega \to \mathbb{R}$ and $Y : \Omega' \to \mathbb{R}$ be two iid copies. We define on the product space

$$(\Omega \times \Omega', \mathcal{A} \otimes \mathcal{A}', \mathbb{P} \times \mathbb{P}')$$

the new random variables $\xi(\omega, \omega') := X(\omega)$ and $\eta(\omega, \omega') := Y(\omega')$. Then we have

- ξ, η live on the same probability space
- $\xi \sim X$, $\eta \sim Y$

$$
\begin{aligned}
\mathbb{P} \times \mathbb{P}'(\xi \in A) &= \mathbb{P} \times \mathbb{P}'(\{(\omega, \omega') \in \Omega \times \Omega' : \xi(\omega, \omega') \in A\}) \\
&= \mathbb{P} \times \mathbb{P}'(\{(\omega, \omega') \in \Omega \times \Omega' : X(\omega) \in A\}) \\
&= \mathbb{P} \times \mathbb{P}'(\{\omega \in \Omega : X(\omega) \in A\} \times \Omega') \\
&= \mathbb{P}(\{\omega \in \Omega : X(\omega) \in A\}) \\
&= \mathbb{P}(X \in A).
\end{aligned}
$$

  and a similar argument works for η.

- $\xi \perp\!\!\!\perp \eta$

$$
\begin{aligned}
\mathbb{P} \times \mathbb{P}'(\xi \in A, \eta \in B) &= \mathbb{P} \times \mathbb{P}'(\{(\omega, \omega') \in \Omega \times \Omega' : \xi(\omega, \omega') \in A, \eta(\omega, \omega') \in B\}) \\
&= \mathbb{P} \times \mathbb{P}'(\{(\omega, \omega') \in \Omega \times \Omega' : X(\omega) \in A, Y(\omega') \in B\}) \\
&= \mathbb{P} \times \mathbb{P}'(\{\omega \in \Omega : X(\omega) \in A\} \times \{\omega \in \Omega' : Y(\omega') \in B\}) \\
&= \mathbb{P}(\{\omega \in \Omega : X(\omega) \in A\}) \, \mathbb{P}'(\{\omega \in \Omega' : Y(\omega') \in B\}) \\
&= \mathbb{P}(X \in A) \, \mathbb{P}(Y \in B) \\
&= \mathbb{P} \times \mathbb{P}'(\xi \in A) \, \mathbb{P} \times \mathbb{P}'(\eta \in B)
\end{aligned}
$$

  The same type of argument works for arbitrary products, since independence is always defined for any finite-dimensional subfamily. In the infinite case, we have to invoke the theorem on the existence of infinite product measures (which are constructed via their finite marginals) and which can be seen as a particular case of Kolmogorov's theorem, cf. Theorem 4.8 and Theorem A.2 in the appendix.

c) The statements are the same if one uses the same construction as above. A difficulty is to identify a multidimensional distribution function $F(x)$. Roughly speaking, these are functions of the form

$$F(x) = \mathbb{P}\Big(X \in (-\infty, x_1] \times \cdots \times (-\infty, x_n]\Big)$$

where $X = (X_1, \ldots, X_n)$ and $x = (x_1, \ldots, x_n)$, i.e. x is the 'upper right' endpoint of an infinite rectancle. An abstract characterisation is the following

- $F : \mathbb{R}^n \to [0,1]$
- $x_j \mapsto F(x)$ is monotone increasing
- $x_j \mapsto F(x)$ is right continuous
- $F(x) = 0$ if at least one entry $x_j = -\infty$
- $F(x) = 1$ if all entries $x_j = +\infty$
- $\sum(-1)^{\sum_{k=1}^n \epsilon_k} F\big(\epsilon_1 a_1 + (1-\epsilon_1)b_1, \dots, \epsilon_n a_n + (1-\epsilon_n)b_n\big) \geqslant 0$ where $-\infty < a_j < b_j < \infty$ and where the outer sum runs over all tuples $(\epsilon_1, \dots, \epsilon_n) \in \{0,1\}^n$

The last property is equivalent to

- $\Delta_{h_1}^{(1)} \cdots \Delta_{h_n}^{(n)} F(x) \geqslant 0 \quad \forall h_1, \dots, h_n \geqslant 0$ where $\Delta_h^{(k)} F(x) = F(x + he_k) - F(x)$ and e_k is the kth standard unit vector of $\mathbb{R}^n$.

In principle we can construct such a multidimensional F from its marginals using the theory of copulas, in particular, Sklar's theorem etc. etc. etc.

Another way would be to take $(\Omega, \mathcal{A}, \mathbb{P}) = (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \mu)$ where μ is the probability measure induced by $F(x)$. Then the random variables X_n are just the identity maps! The independent copies are then obtained by the usual product construction.

Problem 4.2 (Solution) *Step 1:* Let us first show that $\mathbb{P}(\lim_{s\to t} X_s \ \text{ exists}) < 1$.

Since $X_r \perp\!\!\!\perp X_s$ and $X_s \sim -X_s$ we get

$$X_r - X_s \sim X_r + X_s \sim \mathsf{N}(0, s+r) \sim \sqrt{s+r}\,\mathsf{N}(0,1).$$

Thus,

$$\mathbb{P}(|X_r - X_s| > \epsilon) = \mathbb{P}\left(|X_1| > \frac{\epsilon}{\sqrt{s+r}}\right) \xrightarrow[r,s\to t]{} \mathbb{P}\left(|X_1| > \frac{\epsilon}{\sqrt{2t}}\right) \neq 0.$$

This proves that X_s is not a Cauchy sequence in probability, i. e. it does not even converge **in probability** towards a limit, so a.e. convergence is impossible.

In fact we have

$$\left\{\omega : \lim_{s\to t} X_s(\omega) \text{ does not exist}\right\} \supset \bigcap_{k=1}^{\infty} \left\{ \sup_{s,r\in[t-1/k,t+1/k]} |X_s - X_r| > 0 \right\}$$

and so we find with the above calculation

$$\mathbb{P}\left(\lim_{s\to t} X_s \text{ does not exist}\right) \geqslant \lim_k \mathbb{P}\left(\sup_{s,r\in[t-1/k,t+1/k]} |X_s - X_r| > 0\right) \geqslant \mathbb{P}\left(|X_1| > \frac{\epsilon}{\sqrt{2t}}\right)$$

This shows, in particular that for any sequence $t_n \to t$ we have

$$\mathbb{P}\left(\lim_{n\to\infty} X_{t_n} \ \text{ exists}\right) < q < 1.$$

where $q = q(t)$ (but independent of the sequence).

Step 2: Fix $t > 0$, fix a sequence $(t_n)_n$ with $t_n \to t$, and set

$$A = \{\omega \in \Omega : \lim_{s\to t} X_s(\omega) \ \text{ exists}\} \quad \text{and} \quad A(t_n) = \{\omega \in \Omega : \lim_{n\to\infty} X_{t_n}(\omega) \ \text{ exists}\}.$$

Clearly, $A \subset A(t_n)$ for any such sequence. Moreover, take two sequences $(s_n)_n, (t_n)_n$ such that $s_n \to t$ and $t_n \to t$ and which have no points in common; then we get by independence and step 1

$$(X_{s_1}, X_{s_2}, X_{s_3} \dots) \perp\!\!\!\perp (X_{t_1}, X_{t_2}, X_{t_3} \dots) \implies A(t_n) \perp\!\!\!\perp A(s_n)$$

and so, $\mathbb{P}(A) \leqslant \mathbb{P}(A(s_n) \cap A(t_n)) = \mathbb{P}(A(s_n))\,\mathbb{P}(A(t_n)) = q^2$.

By Step 1, $q < 1$. Since there are infinitely many sequences having all no points in common, we get $0 \leqslant \mathbb{P}(A) \leqslant \lim_{k\to\infty} q^k = 0$.

5 Brownian Motion as a Martingale

Problem 5.1 (Solution) a) We have

$$\mathcal{F}_t^B \subset \sigma(\sigma(X), \mathcal{F}_t^B) = \sigma(X, B_s : s \leqslant t) = \widetilde{\mathcal{F}}_t.$$

Let $s \leqslant t$. Then $\sigma(B_t - B_s)$, $\mathcal{F}_s^B$ and $\sigma(X)$ are independent, thus $\sigma(B_t - B_s)$ is independent of $\sigma(\sigma(X), \mathcal{F}_t^B) = \widetilde{\mathcal{F}}_t$. This shows that $\mathcal{F}_t^B$ is an admissible filtration for $(B_t)_{t\geqslant 0}$.

b) Set $\mathcal{N} := \{N : \exists M \in \mathcal{A} \text{ such that } N \subset M, \mathbb{P}(M) = 0\}$. Then we have

$$\mathcal{F}_t^B \subset \sigma(\mathcal{F}_t^B, \mathcal{N}) = \overline{\mathcal{F}}_t^B.$$

From measure theory we know that $(\Omega, \mathcal{A}, \mathbb{P})$ can be completed to $(\Omega, \mathcal{A}^*, \mathbb{P}^*)$ where

$$\mathcal{A}^* := \{A \cup N : A \in \mathcal{A}, N \in \mathcal{N}\},$$
$$\mathbb{P}^*(A^*) := \mathbb{P}(A) \quad \text{for} \quad A^* = A \cup N \in \mathcal{A}^*.$$

We find for all $A \in \mathcal{B}(\mathbb{R}^d)$, $F \in \mathcal{F}_s$, $N \in \mathcal{N}$

$$\begin{aligned}\mathbb{P}^*(\{B_t - B_s \in A\} \cap (F \cup N)) &= \mathbb{P}^*\Big(\underbrace{(\{B_t - B_s \in A\} \cap F)}_{\in \mathcal{A}} \cup \underbrace{(\{B_t - B_s \in A\} \cap N)}_{\in \mathcal{N}}\Big)\\ &= \mathbb{P}(\{B_t - B_s \in A\} \cap F)\\ &= \mathbb{P}(B_t - B_s \in A)\,\mathbb{P}(F)\\ &= \mathbb{P}^*(B_t - B_s \in A)\,\mathbb{P}^*(F \cup N).\end{aligned}$$

Therefore $\overline{\mathcal{F}}_t^B$ is admissible.

Problem 5.2 (Solution) Let $t = t_0 < \ldots < t_n$, and consider the random variables

$$B(t_1) - B(t_0), \ldots, B(t_n) - B(t_{n-1}).$$

Using the argument of Problem 2.9 we see for any $F \in \mathcal{F}_t$

$$\begin{aligned}\mathbb{E}\left(e^{i\sum_{k=1}^n \langle \xi_k, B(t_k) - B(t_{k-1})\rangle} \mathbb{1}_F\right) &= \mathbb{E}\Big(e^{i\langle \xi_n, B(t_n) - B(t_{n-1})\rangle} \cdot \underbrace{e^{i\sum_{k=1}^{n-1} \langle \xi_k, B(t_k) - B(t_{k-1})\rangle} \mathbb{1}_F}_{\mathcal{F}_{t_{n-1}} \text{ mble., hence } \perp\!\!\!\perp B(t_n) - B(t_{n-1})}\Big)\\ &= \mathbb{E}\left(e^{i\langle \xi_n, B(t_n) - B(t_{n-1})\rangle}\right) \cdot \mathbb{E}\left(e^{i\sum_{k=1}^{n-1} \langle \xi_k, B(t_k) - B(t_{k-1})\rangle} \mathbb{1}_F\right)\\ &\;\;\vdots\end{aligned}$$

$$= \prod_{k=1}^{n} \mathbb{E}\left(e^{i\langle \xi_k,\, B(t_k)-B(t_{k-1})\rangle}\right) \mathbb{E}\,\mathbb{1}_F.$$

This shows that the increments are independent among themselves (use $F = \Omega$) and that they are all together independent of $\mathcal{F}_t$ (use the above calculation and the fact that the increments are among themselves independent to combine again the $\prod_1^n$ under the expected value)

Thus,

$$\mathcal{F}_t \perp\!\!\!\perp \sigma\big(B(t_k) - B(t_{k-1}) \,:\, k = 1,\ldots,n\big)$$

Therefore the statement is implied by

$$\mathcal{F}_t \perp\!\!\!\perp \bigcup_{\substack{t<t_1<\ldots<t_n\\ n\geqslant 1}} \sigma\big(B(t_k) - B(t) \,:\, k = 1,\ldots,n\big).$$

Problem 5.3 (Solution) a) i) $\mathbb{E}|X_t| < \infty$, since the expectation does not depend on the filtration.

ii) X_t is $\mathcal{F}_t$ measurable and $\mathcal{F}_t \subset \mathcal{F}_t^*$. Thus X_t is $\mathcal{F}_t^*$ measurable.

iii) Let $\mathcal{N}$ denote the set of all sets which are subsets of $\mathbb{P}$-null sets. Denote by $\mathbb{P}^*$ the measure of the completion of $(\Omega, \mathcal{A}, \mathbb{P})$ (compare with the solution to Exercise 5.1.b)).

Let $t \geqslant s$. For all $F^* \in \mathcal{F}_s^*$ there exist $F \in \mathcal{F}_s$, $N \in \mathcal{N}$ such that $F^* = F \cup N$ and

$$\int_{F^*} X_s \, d\mathbb{P}^* = \int_F X_s \, d\mathbb{P} = \int_F X_t \, d\mathbb{P} = \int_{F^*} X_t \, d\mathbb{P}^*\,.$$

Since F^* is arbitrary this implies that $\mathbb{E}(X_t \,|\, F_s^*) = X_s$.

b) i) $\mathbb{E}|Y_t| = \mathbb{E}|X_t| < \infty$.

ii) Note that $\{X_t \neq Y_t\}$, its complement and any of its subsets is in $\mathcal{F}_t^*$. Let $B \in \mathcal{B}(\mathbb{R}^d)$. Then we get

$$\{Y_t \in B\} = (\underbrace{\{X_t \in B\}}_{\in\mathcal{F}_t} \cap \underbrace{\{X_t \neq Y_t\}^c}_{\in\mathcal{F}_t^*}) \cup \underbrace{\{Y_t \in B, X_t \neq Y_t\}}_{\in\mathcal{F}_t^*}.$$

iii) Similar to part a-iii). For each $F^* \in \mathcal{F}_s^*$ we get

$$\int_{F^*} Y_s \, d\mathbb{P}^* = \int_{F^*} X_s \, d\mathbb{P}^* \overset{\text{a)}}{=} \int_{F^*} X_t \, d\mathbb{P}^* = \int_{F^*} Y_t \, d\mathbb{P}^*,$$

i.e. $\mathbb{E}(Y_t \,|\, \mathcal{F}_s^*) = Y_s$.

Problem 5.4 (Solution) Let $s < t$ and pick $s_n \downarrow s$ such that $s < s_n < t$. Then

$$\mathbb{E}(X_t \mid \mathcal{F}_{s+}) \xleftarrow[s_n\downarrow s]{\text{sub-MG}} \mathbb{E}(X(t) \mid \mathcal{F}_{s_n}) \geqslant X(s_n) \xrightarrow[n\to\infty]{\text{a.e.}} X(s+) \underset{\text{paths}}{\overset{\text{continuous}}{=}} X(s).$$

The convergence on the left side follows from the (sub-)martingale convergence theorem (Lévy's downward theorem).

Problem 5.5 (Solution) Here is a direct proof without using the hint.

We start with calculating the conditional expectations

$$\begin{aligned}
&\mathbb{E}\big(B_t^4 \,\big|\, \mathcal{F}_s\big)\\
&\quad= \mathbb{E}\left((B_t - B_s + B_s)^4 \,\middle|\, \mathcal{F}_s\right)\\
&\quad= B_s^4 + 4B_s^3\,\mathbb{E}(B_t - B_s) + 6B_s^2\,\mathbb{E}((B_t - B_s)^2) + 4B_s\,\mathbb{E}((B_t - B_s)^3) + \mathbb{E}((B_t - B_s)^4)\\
&\quad= B_s^4 + 6B_s^2(t-s) + 3(t-s)^2\\
&\quad= B_s^4 - 6B_s^2 s + 6B_s^2 t + 3(t-s)^2,
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\big(B_t^2 \,\big|\, \mathcal{F}_s\big) = \mathbb{E}\left((B_t - B_s + B_s)^2 \,\middle|\, \mathcal{F}_s\right)\\
&\quad= t - s + 2B_s\,\mathbb{E}(B_t - B_s) + B_s^2\\
&\quad= B_s^2 + t - s.
\end{aligned}$$

Combining these calculations, such that the term $6B_s^2 t$ vanishes from the first formula, we get

$$\begin{aligned}
\mathbb{E}\left(B_t^4 - 6tB_t^2 \,\middle|\, \mathcal{F}_s\right) &= B_s^4 - 6sB_s^2 - 6t^2 + 6st + 3t^2 - 6st + 3s^2\\
&= B_s^4 - 6sB_s + 3s^2 - 3t^2.
\end{aligned}$$

Therefore $\pi(t, B_t) := B_t^4 - 6tB_t^2 + 3t^2$ is a martingale.

Problem 5.6 (Solution) For $t = 0$ and all c we have

$$\mathbb{E}\,e^{c|B_0|} = \mathbb{E}\,e^{c|B_0|^2} = 1.$$

and for $c \leqslant 0$

$$\mathbb{E}\,e^{c|B_0|} \leqslant 1 \quad \text{and} \quad \mathbb{E}\,e^{c|B_0|^2} \leqslant 1.$$

Now let $t > 0$ and $c > 0$. There exists some $R > 0$ such that $c|x| < \frac{1}{4t}|x|^2$ for all $|x| > R$. Thus

$$\begin{aligned}
\mathbb{E}\,e^{c|B_t|} &= \tilde{c}\int e^{c|x|}e^{-\frac{1}{2t}|x|^2}\,dx\\
&\leqslant \tilde{c}\int_{|x|\leqslant R} e^{c|x|}e^{-\frac{1}{2t}|x|^2}\,dx + \tilde{c}\int_{|x|>R} e^{\frac{1}{4t}|x|^2}\,e^{-\frac{1}{2t}|x|^2}\,dx\\
&\leqslant e^{cR} + \tilde{c}\int_{|x|>R} e^{-\frac{1}{4t}|x|^2}\,dx < \infty,
\end{aligned}$$

i.e., $\mathbb{E}\,e^{c|B_t|} < \infty$ for all c, t. Furthermore

$$\mathbb{E}\,e^{c|B_t|^2} = \tilde{c}\int e^{c|x|^2 - \frac{1}{2t}|x|^2}\,dx = \tilde{c}\int e^{|x|^2(c - \frac{1}{2t})}\,dx$$

and this integral is finite if, and only if, $c - \frac{1}{2t} < 0$ or equivalently $c < \frac{1}{2t}$.

Problem 5.7 (Solution) a) We have $p(t,x) = (2\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2t}}$. By the chain rule we get

$$\frac{\partial}{\partial t}p(t,x) = -\frac{d}{2}\,t^{-\frac{d}{2}-1}(2\pi)^{-\frac{d}{2}}e^{-\frac{|x|^2}{2t}} + (2\pi t)^{-\frac{d}{2}}(-1)t^{-2}\,(-1)\frac{|x|^2}{2}\,e^{-\frac{|x|^2}{2t}}$$

and for all $j = 1,\dots,d$

$$\begin{aligned}
\frac{\partial}{\partial x_j}p(t,x) &= (2\pi t)^{-\frac{d}{2}}\Big(-\frac{2x_j}{2t}\Big)\,e^{-\frac{|x|^2}{2t}},\\
\frac{\partial^2}{\partial x_j^2}p(t,x) &= (2\pi t)^{-\frac{d}{2}}\Big(-\frac{1}{t}\Big)\,e^{-\frac{|x|^2}{2t}} + (2\pi t)^{-\frac{d}{2}}\,\frac{x_j^2}{t^2}\,e^{-\frac{|x|^2}{2t}}.
\end{aligned}$$

Adding these terms and noting that $|x|^2 = \sum_{j=1}^d x_j^2$ we get

$$\frac{1}{2}\sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}p(t,x) = -\frac{d}{2}(2\pi t)^{-\frac{d}{2}}t^{-1}e^{-\frac{|x|^2}{2t}} + \frac{(2\pi t)^{-\frac{d}{2}}}{2}\,\frac{|x|^2}{t^2}\,e^{-\frac{|x|^2}{2t}} = \frac{\partial}{\partial t}p(t,x).$$

b) A formal calculation yields

$$\begin{aligned}
&\int p(t,x)\,\frac{1}{2}\frac{\partial^2}{\partial x_j^2}f(t,x)\,dx\\
&= p(t,x)\,\frac{1}{2}\frac{\partial}{\partial x_j}f(t,x)\Big|_{-\infty}^{\infty} - \int \frac{\partial}{\partial x_j}p(t,x)\cdot\frac{1}{2}\frac{\partial}{\partial x_j}f(t,x)\,dx\\
&= 0 - \frac{\partial}{\partial x_j}p(t,x)\cdot\frac{1}{2}f(t,x)\Big|_{-\infty}^{\infty} + \int \frac{\partial^2}{\partial x_j^2}p(t,x)\cdot\frac{1}{2}f(t,x)\,dx\\
&= \int \frac{\partial^2}{\partial x_j^2}p(t,x)\cdot\frac{1}{2}f(t,x)\,dx.
\end{aligned}$$

By the same arguments as in Exercise 5.6 we find that all terms are integrable and vanish as $|x|\to\infty$. This justifies the above calculation. Furthermore summing over $j = 1,\dots d$ we obtain the statement.

Problem 5.8 (Solution) Measurability (i.e. adaptedness to the Filtration $\mathcal{F}_t$) and integrability is no issue, see also Problem 5.6.

a) U_t is only a martingale for $c = 0$.

Solution 1: see Exercise 5.9.

Solution 2: if $c \neq 0$, $\mathbb{E}\,U_t$ is not constant, i.e. cannot be a martingale. If $c = 0$, U_t is trivially a martingale.

b) V_t is a martingale since

$$\begin{aligned}
\mathbb{E}\big(V_t\,\big|\,\mathcal{F}_s\big) &= t\,\mathbb{E}(B_t - B_s) + tB_s - \mathbb{E}\bigg(\int_0^s B_r\,dr\,\bigg|\,\mathcal{F}_s\bigg) - \mathbb{E}\bigg(\int_s^t B_r\,dr\,\bigg|\,\mathcal{F}_s\bigg)\\
&= tB_s - \int_0^s B_r\,dr - \mathbb{E}\bigg(\int_s^t (B_r - B_s) + B_s\,dr\,\bigg|\,\mathcal{F}_s\bigg)\\
&= tB_s - \int_0^s B_r\,dr - (t-s)B_s\\
&= V_s.
\end{aligned}$$

c) and e) Let $a \in \mathbb{R}$. Then we get

$$\begin{aligned}\mathbb{E}\left(aB_t^3 - tB_t \,\middle|\, \mathcal{F}_s\right) &= \mathbb{E}\left(a(B_t - B_s + B_s)^3 - t(B_t - B_s) - tB_s \,\middle|\, \mathcal{F}_s\right)\\ &= aB_s^3 + 3aB_s^2\,\mathbb{E}\,B_{t-s} + 3aB_s\,\mathbb{E}\,B_{t-s}^2 + a\,\mathbb{E}\,B_{t-s}^3 - 0 - tB_s\\ &= aB_s^3 + (3a(t-s) - t)B_s.\end{aligned}$$

This is a martingale if, and only if, $-s = 3a(t-s) - t$, i.e., $a = \frac{1}{3}$. Thus Y_t is a martingale and W_t is not a martingale.

d) We have seen in part c) and b) that

$$\mathbb{E}\left(B_t^3 \,\middle|\, \mathcal{F}_s\right) = B_s^3 + 3(t-s)B_s$$

and

$$3\,\mathbb{E}\left(\int_0^t B_r\,dr \,\middle|\, \mathcal{F}_s\right) = 3\int_0^s B_r\,dr + 3(t-s)B_s.$$

Thus, X_t is a martingale.

f) Z_t is only a martingale for $c = \frac{1}{2}$, see Exercise 5.9.

Problem 5.9 (Solution) Note that $\mathbb{E}\,|X_t| < \infty$ for all a, b, cf. Problem 5.6. We have

$$\begin{aligned}\mathbb{E}\left(e^{aB_t + bt} \,\middle|\, \mathcal{F}_s\right) &= \mathbb{E}\left(e^{a(B_t - B_s)} e^{aB_s + bt} \,\middle|\, \mathcal{F}_s\right)\\ &= e^{aB_s + bt}\,\mathbb{E}\,e^{aB_{t-s}}\\ &= e^{aBs + bt + (t-s)a^2/2}.\end{aligned}$$

Thus, X_t is a martingale if, and only if, $bs = bt + (t-s)\frac{a^2}{2}$, i.e., $b = -\frac{1}{2}\,a^2$.

Problem 5.10 (Solution) We have

$$\mathbb{E}\left(\tfrac{1}{d}\,|B_t|^2 - t \,\middle|\, \mathcal{F}_s\right) = -t + \tfrac{1}{d}\sum_{j=1}^d \mathbb{E}\left(\left(B_t^{(j)}\right)^2 \,\middle|\, \mathcal{F}_s\right) \overset{\text{Pr. 5.5}}{=} -t + \tfrac{1}{d}\sum_{j=1}^d \left(\left(B_s^{(j)}\right)^2 + t - s\right) = \tfrac{1}{d}\,|B_s|^2 - s.$$

Problem 5.11 (Solution) For a)–c) we prove only the statements for τ°, the statements for τ are proved analogously.

a) The following implications hold:

$$A \subset C \implies \{t \geqslant 0 : X_t \in A\} \subset \{t \geqslant 0 : X_t \in C\} \implies \tau_A^\circ \geqslant \tau_C^\circ.$$

b) By part a) we have $\tau_{A\cup C}^\circ \leqslant \tau_A^\circ$ and $\tau_{A\cup C}^\circ \leqslant \tau_C^\circ$. Thus,

$$\tau_{A\cup C}^\circ \overset{a)}{\leqslant} \min\{\tau_A^\circ, \tau_C^\circ\}.$$

To see the converse, $\min\{\tau_A^\circ, \tau_C^\circ\} \leqslant \tau_{A\cup C}^\circ$, it is enough to show that

$$X_t(\omega) \in A \cup C \implies t \geqslant \min\{\tau_A^\circ(\omega), \tau_C^\circ(\omega)\}$$

since this implication shows that $\tau_{A\cup C}^\circ(\omega) \geqslant \min\{\tau_A^\circ(\omega), \tau_C^\circ(\omega)\}$ holds.

Now observe that

$$\begin{aligned}
X_t(\omega) \in A \cup C &\implies X_t(\omega) \in A \quad \text{or} \quad X_t(\omega) \in C \\
&\implies t \geqslant \tau_A^\circ(\omega) \quad \text{or} \quad t \geqslant \tau_C^\circ(\omega) \\
&\implies t \geqslant \min\{\tau_A^\circ(\omega), \tau_C^\circ(\omega)\}.
\end{aligned}$$

c) Part a) implies $\max\{\tau_A^\circ, \tau_C^\circ\} \leqslant \tau_{A\cap C}^\circ$.

Remark: we cannot expect "=". To see this consider a BM[1] staring at $B_0 = 0$ and the set

$$A = [4,6] \quad \text{and} \quad C = [1,2] \cup [5,7].$$

Then B_t has to reach first C and A before it hits $A \cap C$.

d) as in b) it is clear that $\tau_A^\circ \leqslant \tau_{A_n}^\circ$ for all $n \geqslant 1$, hence

$$\tau_A^\circ \leqslant \inf_{n\geqslant 1} \tau_{A_n}^\circ.$$

In order to show the converse, $\tau_A^\circ \geqslant \inf_{n\geqslant 1} \tau_{A_n}^\circ$, it is enough to check that

$$X_t(\omega) \in A \implies t \geqslant \inf_{n\geqslant 1} \tau_{A_n}^\circ(\omega)$$

since, if this is true, this implies that $\tau_A^\circ(\omega) \geqslant \inf_{n\geqslant 0} \tau_{A_n}^\circ(\omega)$.

Now observe that

$$\begin{aligned}
X_t(\omega) \in A = \cup_n A_n &\implies X_t(\omega) \in A_n \text{ for some } n \in \mathbb{N} \\
&\implies t \geqslant \tau_{A_n}^\circ(\omega) \text{ for some } n \in \mathbb{N} \\
&\implies t \geqslant \inf_{n\geqslant 0} \tau_{A_n}^\circ(\omega).
\end{aligned}$$

e) Note that $\inf\left\{s \geqslant 0 \,:\, X_{s+\frac{1}{n}} \in A\right\} = \inf\left\{s \geqslant \frac{1}{n} \,:\, X_s \in A\right\}$ is monotone decreasing as $n \to \infty$. Thus we get

$$\begin{aligned}
\inf_n \left(\tfrac{1}{n} + \inf\{s \geqslant \tfrac{1}{n} \,:\, X_s \in A\}\right) &= 0 + \inf_n \inf\{s \geqslant \tfrac{1}{n} \,:\, X_s \in A\} \\
&= \inf\{s > 0 \,:\, X_s \in A\} \\
&= \tau_A.
\end{aligned}$$

f) Let $X_t = x_0 + t$. Then $\tau_{\{x_0\}}^\circ = 0$ and $\tau_{\{x_0\}} = \infty$.

More generally, a similar situation may happen if we consider a process with continuous paths, a closed set F, and if we let the process start on the boundary ∂F. Then $\tau_F^\circ = 0$ a.s. (since the process is in the set) while $\tau_F > 0$ is possible with positive probability.

Problem 5.12 (Solution) We have $\tau_U^\circ \leqslant \tau_U$.

Let $x_0 \in U$. Then $\tau_U^\circ = 0$ and, since U is open and X_t is continuous, there exists an $N > 0$ such that

$$X_{\frac{1}{n}} \in U \quad \text{for all } n \geqslant N.$$

Thus $\tau_U = 0$.

If $x_0 \notin U$, then $X_t(\omega) \in U$ can only happen if $t > 0$. Thus, $\tau_U^\circ = \tau_U$.

Problem 5.13 (Solution) Suppose $d(x, A) \geqslant d(z, A)$. Then

$$\begin{aligned} d(x,A) - d(z,A) &= \inf_{y \in A} |x-y| - \inf_{y \in A} |z-y| \\ &\leqslant \inf_{y \in A} (|x-z| + |z-y|) - \inf_{y \in A} |z-y| \\ &= |x-z| \end{aligned}$$

and, with an analogous argument for $d(x, A) \leqslant d(z, A)$, we conclude

$$|d(x,A) - d(z,A)| \leqslant |x-z|.$$

Thus $x \mapsto d(x, A)$ is globally Lipschitz continuous, hence uniformly continuous.

Problem 5.14 (Solution) We treat the two cases simultaneously and check the three properties of a sigma algebra:

i) We have $\Omega \in \mathcal{F}_\infty$ and

$$\Omega \cap \{\tau \leqslant t\} = \{\tau \leqslant t\} \in \mathcal{F}_t \subset \mathcal{F}_{t+}.$$

ii) Let $A \in \mathcal{F}_{\tau(+)}$. Thus $A \in \mathcal{F}_\infty$, $A^c \in \mathcal{F}_\infty$ and

$$A^c \cap \{\tau \leqslant t\} = \Omega \setminus A \cap \{\tau \leqslant t\} = \underbrace{(\Omega \cap \{\tau \leqslant t\})}_{\in \mathcal{F}_t \subset \mathcal{F}_{t+}} \setminus \underbrace{(A \cap \{\tau \leqslant t\})}_{\in \mathcal{F}_{t(+)} \text{ since } A \in \mathcal{F}_{\tau(+)}} \in \mathcal{F}_{t(+)}.$$

iii) Let $A_n \in \mathcal{F}_{\tau(+)}$. Then $A_n, \bigcup_n A_n \in \mathcal{F}_\infty$ and

$$\bigcup_n A_n \cap \{\tau \leqslant t\} = \bigcup_n \underbrace{(A_n \cap \{\tau \leqslant t\})}_{\in \mathcal{F}_{t(+)}} \in \mathcal{F}_{t(+)}.$$

Therefore $\mathcal{F}_\tau$ and $\mathcal{F}_{\tau+}$ are σ-algebras.

Problem 5.15 (Solution) a) Let $F \in \mathcal{F}_{\tau+}$, i.e., $F \in \mathcal{F}_\infty$ and for all s we have $F \cap \{\tau \leqslant s\} \in \mathcal{F}_{s+}$.

Let $t > 0$. Then

$$F \cap \{\tau < t\} = \bigcup_{s<t} (F \cap \{\tau \leqslant s\}) \in \bigcup_{s<t} \mathcal{F}_{s+} \subset \mathcal{F}_t.$$

For the converse: If $\tau < \infty$ a.s. then $F = \bigcup_{t>0}(F \cap \{\tau \leqslant t\}) \in \mathcal{F}_\infty$ and

$$F \cap \{\tau \leqslant s\} = \bigcap_{t>s} (F \cap \{\tau < t\}) \in \bigcap_{t>s} \mathcal{F}_t = \mathcal{F}_{s+}.$$

If $\tau = \infty$ occurs with strictly positive probability, then we have to assume that $F \in \mathcal{F}_\infty$.

b) We have $\{\tau \leqslant t\} \in \mathcal{F}_t \subset \mathcal{F}_\infty$ and

$$\{\tau \leqslant t\} \cap \{\tau \wedge t \leqslant r\} = \begin{cases} \{\tau \leqslant t\} \in \mathcal{F}_t & \text{if } r \geqslant t; \\ \{\tau \leqslant r\} \in \mathcal{F}_r \subset \mathcal{F}_t & \text{if } r < t. \end{cases}$$

Problem 5.16 (Solution) a) $e^{i\xi B_t + \frac{1}{2} t|\xi|^2}$ is a martingale for all $\xi \in \mathbb{R}$ by Example 5.2 d). By optional stopping

$$1 = \mathbb{E}\, e^{\frac{1}{2}(\tau\wedge t)c^2 + icB_{\tau\wedge t}}.$$

Since the left-hand side is real, we get

$$1 = \mathbb{E}\left(e^{\frac{1}{2}(\tau\wedge t)c^2} \cos(cB_{\tau\wedge t})\right).$$

Set $m := a \vee b$. Since $|B_{\tau\wedge t}| \leqslant m$, we see that for $mc < \frac{1}{2}\pi$ the cosine is positive. By Fatou's lemma we get for all $mc < \frac{1}{2}\pi$

$$\begin{aligned} 1 &= \varliminf_{t\to\infty} \mathbb{E}\left(e^{\frac{1}{2}(\tau\wedge t)c^2} \cos(cB_{\tau\wedge t})\right) \\ &\geqslant \mathbb{E}\left(\varliminf_{t\to\infty} e^{\frac{1}{2}(\tau\wedge t)c^2} \cos(cB_{\tau\wedge t})\right) \\ &\geqslant \mathbb{E}\left(e^{\frac{1}{2}\tau c^2} \cos(cB_\tau)\right) \\ &\geqslant \cos(mc)\, \mathbb{E}\, e^{\frac{1}{2}\tau c^2}. \end{aligned}$$

Thus, $\mathbb{E}\, e^{\gamma\tau} < \infty$ for any $\gamma < \frac{1}{2}c^2$ and all $c < \pi/(2m)$. Since

$$e^t = \sum_{j=0}^\infty \frac{t^j}{j!} \implies \forall t \geqslant 0,\ j \geqslant 0 \ :\ e^t \geqslant \frac{t^j}{j!}$$

we see that $\mathbb{E}\,\tau^j \leqslant j!\,\gamma^{-j}\, \mathbb{E}\, e^{\gamma\tau} < \infty$ for any $j \geqslant 0$.

b) By Exercise 5.8 d) the process $B_t^3 - 3\int_0^t B_s\,ds$ is a martingale. By optional stopping we get

$$\mathbb{E}\left(B_{\tau\wedge t}^3 - 3\int_0^{\tau\wedge t} B_s\,ds\right) = 0 \quad \text{for all} \ \ t \geqslant 0. \tag{*}$$

Set $m = \max\{a, b\}$. By the definition of τ we see that $|B_{\tau\wedge t}| \leqslant m$; since τ is integrable we get

$$|B_{\tau\wedge t}^3| \leqslant m^3 \quad \text{and} \quad \left|\int_0^{\tau\wedge t} B_s\,ds\right| \leqslant \tau\cdot m.$$

Therefore, we can use in (*) the dominated convergence theorem and let $t \to \infty$:

$$\begin{aligned} \mathbb{E}\left(\int_0^\tau B_s\,ds\right) &= \frac{1}{3}\,\mathbb{E}(B_\tau^3) \\ &= \frac{1}{3}(-a)^3\,\mathbb{P}(B_\tau = -a) + \frac{1}{3}\,b^3\,\mathbb{P}(B_\tau = b) \\ &\overset{(5.12)}{=} \frac{1}{3}\,\frac{-a^3b + b^3a}{a+b} \\ &= \frac{1}{3}\,ab(b-a). \end{aligned}$$

Problem 5.17 (Solution) By Example 5.2 c) $|B_t|^2 - d{\cdot}t$ is a martingale. Thus we get by optional stopping

$$\mathbb{E}(t \wedge \tau_R) = \frac{1}{d}\,\mathbb{E}\,|B_{t\wedge\tau_R}|^2 \quad \text{for all} \quad t \geqslant 0.$$

Since $|B_{t\wedge\tau_R}| \leqslant R$, we can use monotone convergence on the left and dominated convergence on the right-hand side to get

$$\mathbb{E}\,\tau_R = \sup_{t\geqslant 0}\mathbb{E}(t\wedge\tau_R) = \lim_{t\to\infty}\frac{1}{d}\,\mathbb{E}\,|B_{t\wedge\tau_R}|^2 = \frac{1}{d}\,\mathbb{E}\,|B_{\tau_R}|^2 = \frac{1}{d}\,R^2.$$

Problem 5.18 (Solution) a) For all t we have

$$\{\sigma\wedge\tau \leqslant t\} = \underbrace{\{\sigma \leqslant t\}}_{\in\mathcal{F}_t} \cup \underbrace{\{\tau\leqslant t\}}_{\in\mathcal{F}_t} \in \mathcal{F}_t.$$

b) For all t we have

$$\begin{aligned}\{\sigma<\tau\}\cap\{\sigma\wedge\tau\leqslant t\} &= \bigcup_{0\leqslant r\in\mathbb{Q}} \left(\{\sigma\leqslant r<\tau\}\cap\{\sigma\wedge\tau\leqslant t\}\right)\\ &= \bigcup_{r\in\mathbb{Q}\cap[0,t]}\Big(\left(\{\sigma\leqslant r\}\cap\{\tau\leqslant r\}^c\right)\cap\{\sigma\wedge\tau\leqslant t\}\Big)\in\mathcal{F}_t.\end{aligned}$$

This shows that $\{\sigma<\tau\}, \{\sigma\geqslant\tau\} = \{\sigma<\tau\}^c \in \mathcal{F}_{\sigma\wedge\tau}$. Since σ and τ play symmetric roles, we get with a similar argument that $\{\sigma>\tau\}, \{\sigma\leqslant\tau\} = \{\sigma>\tau\}^c\in\mathcal{F}_{\sigma\wedge\tau}$, and the claim follows.

c) Since $\tau\wedge\sigma$ is an integrable stopping time, we get from Wald's identities, Theorem 5.10, that

$$\mathbb{E}\,B^2_{\tau\wedge\sigma} = \mathbb{E}(\tau\wedge\sigma) < \infty.$$

Following the hint we get

$$\begin{aligned}\mathbb{E}(B_\sigma B_\tau \mathbb{1}_{\{\sigma\leqslant\tau\}}) &= \mathbb{E}(B_{\sigma\wedge\tau}B_\tau\mathbb{1}_{\{\sigma\leqslant\tau\}})\\ &= \mathbb{E}\left(\,\mathbb{E}(B_{\sigma\wedge\tau}B_\tau\mathbb{1}_{\{\sigma\leqslant\tau\}}\,\middle|\,\mathcal{F}_{\tau\wedge\sigma})\right)\\ &\overset{\text{b)}}{=} \mathbb{E}\left(B_{\sigma\wedge\tau}\mathbb{1}_{\{\sigma\leqslant\tau\}}\,\mathbb{E}\left(B_\tau\,\middle|\,\mathcal{F}_{\tau\wedge\sigma}\right)\right)\\ &\overset{(*)}{=} \mathbb{E}(B^2_{\sigma\wedge\tau}\mathbb{1}_{\{\sigma\leqslant\tau\}}).\end{aligned}$$

(We will discuss the step marked by (*) below.)

With an analogous calculation for $\tau\leqslant\sigma$ we conclude

$$\mathbb{E}(B_\sigma B_\tau) = \mathbb{E}(B_{\sigma\wedge\tau}B_\tau\mathbb{1}_{\{\sigma<\tau\}}) + \mathbb{E}(B_{\sigma\wedge\tau}B_\tau\mathbb{1}_{\{\tau\leqslant\sigma\}}) = \mathbb{E}(B^2_{\sigma\wedge\tau}) = \mathbb{E}\,\sigma\wedge\tau.$$

In the step marked with (*) we used that for *integrable* stopping times σ,τ we have

$$\mathbb{E}(B_\tau \mid \mathcal{F}_{\sigma\wedge\tau}) = B_{\sigma\wedge\tau}.$$

To see this we use optional stopping which gives

$$\mathbb{E}(B_{\tau\wedge k}\mid\mathcal{F}_{\sigma\wedge\tau\wedge k}) = B_{\sigma\wedge\tau\wedge k} \quad \text{for all} \quad k\geqslant 1.$$

This is the same as to say that

$$\int_F B_{\tau\wedge k}\,d\mathbb{P} = \int_F B_{\sigma\wedge\tau\wedge k}\,d\mathbb{P} \quad \text{for all}\quad k \geqslant 1,\ F \in \mathcal{F}_{\sigma\wedge\tau\wedge k}.$$

Since $B_{\tau\wedge k} \xrightarrow[k\to\infty]{} B_\tau$ in $L^2(\mathbb{P})$, see the proof of Theorem 5.10, we get for some fixed $i < k$ because of $\mathcal{F}_{\sigma\wedge\tau\wedge i} \subset \mathcal{F}_{\sigma\wedge\tau\wedge k}$ that

$$\int_F B_\tau\,d\mathbb{P} = \lim_{k\to\infty}\int_F B_{\tau\wedge k}\,d\mathbb{P} = \lim_{k\to\infty}\int_F B_{\sigma\wedge\tau\wedge k}\,d\mathbb{P} = \int_F B_{\sigma\wedge\tau}\,d\mathbb{P} \quad \text{for all}\quad F \in \mathcal{F}_{\sigma\wedge\tau\wedge i}.$$

Let $\rho = \sigma\wedge\tau$ (or any other stopping time). Since $\mathcal{F}_{\rho\wedge k} = \mathcal{F}_\rho \cap \mathcal{F}_k$ we see that $\mathcal{F}_\rho$ is generated by the $\cap$-stable generator $\bigcup_i \mathcal{F}_{\rho\wedge i}$, and (*) follows.

d) From the above and Wald's identity we get

$$\begin{aligned}
\mathbb{E}(|B_\tau - B_\sigma|^2) &= \mathbb{E}(B_\tau^2 - 2B_\tau B_\sigma + B_\sigma^2)\\
&= \mathbb{E}\,\tau - 2\,\mathbb{E}\,\tau\wedge\sigma + \mathbb{E}\,\sigma\\
&= \mathbb{E}(\tau - 2(\tau\wedge\sigma) + \sigma)\\
&= \mathbb{E}\,|\tau-\sigma|.
\end{aligned}$$

In the last step we used the elementary relation

$$(a+b) - 2(a\wedge b) = a\wedge b + a\vee b - 2(a\wedge b) = a\vee b - a\wedge b = |a-b|.$$

6 Brownian Motion as a Markov Process

Problem 6.1 (Solution) We write $g_t(x) = (2\pi t)^{-1/2}\, e^{-x^2/(2t)}$ for the one-dimensional normal density.

a) This follows immediately from our proof of b).

b) Let $u \in \mathcal{B}_b(\mathbb{R})$ and $s, t \geqslant 0$. Then, by the independent and stationary increments property of a Brownian motion

$$\begin{aligned}
\mathbb{E}\, u\big(|B_{t+s}|\,\big|\,\mathcal{F}_s\big) &= \mathbb{E}\, u\big(|(B_{t+s} - B_s) + B_s|\,|\,\mathcal{F}_s\big) \\
&= \mathbb{E}\, u\big(|(B_{t+s} - B_s) + y|\big)\Big|_{y=B_s} \\
&= \mathbb{E}\, u\big(|B_t + y|\big)\Big|_{y=B_s}.
\end{aligned}$$

Since $B \sim -B$ we also get

$$\mathbb{E}\, u\big(|B_{t+s}|\,\big|\,\mathcal{F}_s\big) = \mathbb{E}\, u\big(|B_t + y|\big)\Big|_{y=-B_s} = \mathbb{E}\, u\big(|B_t - y|\big)\Big|_{y=B_s}$$

and, therefore,

$$\begin{aligned}
\mathbb{E}\, u\big(|B_{t+s}|\,\big|\,\mathcal{F}_s\big) &= \frac{1}{2}\left[\mathbb{E}\, u\big(|B_t + y|\big) + \mathbb{E}\, u\big(|B_t - y|\big)\right]_{y=B_s} \\
&= \frac{1}{2}\left[\int_{-\infty}^{\infty} \big(u(|z+y|) + u(|z-y|)\big)\, g_t(z)\, dz\right]_{y=B_s} \\
&= \frac{1}{2}\left[\int_{-\infty}^{\infty} u(|z|)\,\big(g_t(z+y) + g_t(z-y)\big)\, dz\right]_{y=B_s} \\
&= \int_0^{\infty} u(|z|)\,\big(g_t(z+y) + g_t(z-y)\big)\, dz\Big|_{y=B_s}
\end{aligned}$$

here we use that the integrand is even in z

$$= \underbrace{\int_0^{\infty} u(|z|)\,\big(g_t(z+|y|) + g_t(z-|y|)\big)\, dz}_{=:g_{u,s,t+s}(y)\text{—it is independent of } s!}\Bigg|_{y=B_s}$$

since the integrand is also even in y! This shows that

- $\mathbb{E}\, u\big(|B_{t+s}|\,\big|\,\mathcal{F}_s\big)$ is a function of $|B_s|$, i.e. Markovianity.
- $\mathbb{P}^y(|B_t| \in dz) = g_t(z-y) + g_t(z+y)$ for $z, y \geqslant 0$, i.e. the form of the transition function.

Remark: $|B_t|$ is called *reflecting* (also: *reflected*) *Brownian motion.*

c) Set $M_t := \sup_{s\leqslant t} B_s$ for the running maximum, i.e. $Y_t = M_t - B_t$. From the reflection principle, Theorem 6.9 we know that $Y_t \sim |B_t|$. So the guess is that Y and $|B|$ are two Markov processes with the same transition function!

Let $s,t\geqslant 0$ and $u\in\mathcal{B}_b(\mathbb{R})$. We have by the independent and stationary increments property of Brownian motion

$$\begin{aligned}
\mathbb{E}\left(u(Y_{t+s})\,\middle|\,\mathcal{F}_s\right) &= \mathbb{E}\left(u(M_{t+s}-B_{t+s})\,\middle|\,\mathcal{F}_s\right)\\
&= \mathbb{E}\left(u\Big(\max\Big\{\sup_{u\leqslant s} B_r,\ \sup_{0\leqslant u\leqslant t} B_{s+u}\Big\} - B_{t+s}\Big)\,\middle|\,\mathcal{F}_s\right)\\
&= \mathbb{E}\left(u\Big(\max\Big\{\sup_{u\leqslant s}(B_r-B_s)+(B_s-B_{t+s}),\ \sup_{0\leqslant u\leqslant t}(B_{s+u}-B_{s+t})\Big\}\Big)\,\middle|\,\mathcal{F}_s\right)
\end{aligned}$$

and, as $\sup_{u\leqslant s}(B_r-B_s)$ is $\mathcal{F}_s$ measurable and $(B_s-B_{t+s}), \sup_{0\leqslant u\leqslant t}(B_{s+u}-B_{s+t}) \perp\!\!\!\perp \mathcal{F}_s$, we get

$$\begin{aligned}
&= \mathbb{E}\left(u\Big(\max\Big\{y+(B_s-B_{t+s}),\ \sup_{0\leqslant u\leqslant t}(B_{s+u}-B_{s+t})\Big\}\Big)\right)\Bigg|_{y=\sup_{u\leqslant s}(B_r-B_s)}\\
&= \mathbb{E}\left(u\Big(\max\Big\{y-B_t,\ \sup_{0\leqslant u\leqslant t}(B_u-B_t)\Big\}\Big)\right)\Bigg|_{y=Y_s}
\end{aligned}$$

Using time inversion (cf. 2.11) we see that $(B_{t-u}-B_t)_{u\in[0,t]}$ is again a BM[1], and we get $\big(B_t, \sup_{0\leqslant u\leqslant t}(B_u-B_t)\big) \sim \big(-B_t, \sup_{0\leqslant u\leqslant t} B_u)\big)$

$$= \mathbb{E}\left(u\Big(\max\Big\{y+B_t,\ \sup_{0\leqslant u\leqslant t} B_u)\Big\}\Big)\right)\Bigg|_{y=Y_s}.$$

Using Solution 2 of Problem 6.8 we know the joint distribution of $(B_t, \sup_{u\leqslant t} B_u)$:

$$\begin{aligned}
&\mathbb{E}\left(u\Big(\max\Big\{y+B_t,\ \sup_{0\leqslant u\leqslant t} B_u)\Big\}\Big)\right)\\
&\quad= \int_{z=0}^{\infty}\int_{x=-\infty}^{z} u(\max\{y+x,z\})\frac{2}{\sqrt{2\pi t}}\,\frac{2z-x}{t}\,e^{-(2z-x)^2/2t}\,dx\,dz.
\end{aligned}$$

Splitting the integral $\int_{x=-\infty}^{z}$ into two parts $\int_{x=-\infty,y+x\leqslant z}^{z} + \int_{x=-\infty,y+x>z}^{z}$ we get

$$I = \int_{z=0}^{\infty} u(z)\frac{2}{\sqrt{2\pi t}}\underbrace{\int_{x=-\infty}^{z-y}\frac{2z-x}{t}\,e^{-(2z-x)^2/2t}\,dx}_{=e^{-(2z-x)^2/2t}\big|_{-\infty}^{z-y}}\,dz = \frac{2}{\sqrt{2\pi t}}\int_{z=0}^{\infty} u(z)\,e^{-(z+y)^2/2t}\,dz$$

and

$$\begin{aligned}
II &= \frac{2}{\sqrt{2\pi t}}\int_{z=0}^{\infty}\int_{x=-z-y}^{z} u(y+x)\,\frac{2z-x}{t}\,e^{-(2z-x)^2/2t}\,dx\,dz\\
&= \frac{2}{\sqrt{2\pi t}}\int_{x=-y}^{\infty} u(y+x)\underbrace{\int_{z=x}^{x+y}\frac{2z-x}{t}\,e^{-(2z-x)^2/2t}\,dz}_{=-\frac{1}{2}e^{-(2z-x)^2/2t}\big|_{z=x}^{x+y}}\\
dx &= \frac{1}{\sqrt{2\pi t}}\int_{x=-y}^{\infty} u(y+x)\left[e^{-x^2/2t}-e^{-(x+2y)^2/2t}\right]dx
\end{aligned}$$

$$= \frac{1}{\sqrt{2\pi t}} \int_{x=-y}^{\infty} u(\xi) \left[e^{-(\xi-y)^2/2t} - e^{-(\xi+y)^2/2t} \right] d\xi.$$

Finally, adding I and II we end up with

$$\mathbb{E}\Big(u\Big(\max\Big\{ y + B_t,\ \sup_{0\leqslant u\leqslant t} B_u) \Big\} \Big) \Big) = \int_0^\infty u(z) \big(g_t(z+y) + g_t(z-y) \big)\, dz, \quad y \geqslant 0$$

which is the same transition function as in part b).

d) See part c).

Problem 6.2 (Solution) Let $s, t \geqslant 0$. We use the following abbreviations:

$$I_s = \int_0^s B_r\, dr \quad \text{and} \quad M_s = \sup_{u\leqslant s} B_u \quad \text{and} \quad \mathcal{F}_s = \mathcal{F}_s^B.$$

a) Let $f : \mathbb{R}^2 \to \mathbb{R}$ measurable and bounded. Then

$$\begin{aligned}
&\mathbb{E}\left(f(M_{s+t}, B_{s+t}) \,\middle|\, \mathcal{F}_s \right) \\
&\quad = \mathbb{E}\Big(f\Big(\sup_{s\leqslant u\leqslant s+t} B_u \vee M_s,\ (B_{s+t} - B_s) + B_s \Big) \,\Big|\, \mathcal{F}_s \Big) \\
&\quad = \mathbb{E}\Big(f\Big(\Big[B_s + \sup_{s\leqslant u\leqslant s+t} (B_u - B_s) \Big] \vee M_s,\ (B_{s+t} - B_s) + B_s \Big) \,\Big|\, \mathcal{F}_s \Big).
\end{aligned}$$

By the independent increments property of BM we get that the random variables $\sup_{s\leqslant u\leqslant s+t}(B_u - B_s)$, $B_{s+t} - B_s \perp\!\!\!\perp \mathcal{F}_s$ while M_s and B_s are $\mathcal{F}_s$ measurable. Thus, we can treat these groups of random variables separately (see, e.g., Lemma A.3:

$$\begin{aligned}
&\mathbb{E}\left(f(M_{s+t}, B_{s+t}) \,\middle|\, \mathcal{F}_s \right) \\
&\quad = \mathbb{E}\Big(f\Big(\Big[z + \sup_{s\leqslant u\leqslant s+t} (B_u - B_s) \Big] \vee y,\ (B_{s+t} - B_s) + z \Big) \,\Big|\, \mathcal{F}_s \Big) \Big|_{y=M_s, z=B_s} \\
&\quad = \phi(M_s, B_s)
\end{aligned}$$

where

$$\phi(y,z) = \mathbb{E}\Big(f\Big(\Big[z + \sup_{s\leqslant u\leqslant s+t} (B_u - B_s) \Big] \vee y,\ (B_{s+t} - B_s) + z \Big) \,\Big|\, \mathcal{F}_s \Big).$$

b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ measurable and bounded. Then

$$\begin{aligned}
&\mathbb{E}\left(f(I_{s+t}, B_{s+t}) \,\middle|\, \mathcal{F}_s \right) \\
&\quad = \mathbb{E}\Big(f\Big(\int_s^{s+t} B_u\, du + I_s,\ (B_{s+t} - B_s) + B_s \Big) \,\Big|\, \mathcal{F}_s \Big) \\
&\quad = \mathbb{E}\Big(f\Big(\int_s^{s+t} (B_u - B_s)\, du + I_s + tB_s,\ (B_{s+t} - B_s) + B_s \Big) \,\Big|\, \mathcal{F}_s \Big).
\end{aligned}$$

By the independent increments property of BM we get that the random variables $\int_s^{s+t}(B_u - B_s)\, du$, $B_{s+t} - B_s \perp\!\!\!\perp \mathcal{F}_s$ while $I_s + tB_s$ and B_s are $\mathcal{F}_s$ measurable. Thus, we can treat these groups of random variables separately (see, e.g., Lemma A.3:

$$\mathbb{E}\left(f(I_{s+t}, B_{s+t}) \,\middle|\, \mathcal{F}_s \right)$$

$$= \mathbb{E}\Big(f\Big(\int_s^{s+t}(B_u - B_s)\,du + y + tz,\ (B_{s+t} - B_s) + z\Big)\Big)\Big|_{y=I_s, z=B_s}$$
$$= \phi(I_s, B_s)$$

for the function

$$\phi(y,z) = \mathbb{E}\Big(f\Big(\int_s^{s+t}(B_u - B_s)\,du + y + tz,\ (B_{s+t} - B_s) + z\Big)\Big).$$

c) No! If we use the calculation of a) and b) for the function $f(y,z) = g(y)$, i.e. only depending on M or I, respectively, we see that we still get

$$\mathbb{E}\big(g(I_{t+s}) \,\big|\, \mathcal{F}_s\big) = \psi(B_s, I_s),$$

i.e. $(I_t, \mathcal{F}_t)_t$ cannot be a Markov process. The same argument applies to $(M_t, \mathcal{F}_t)_t$.

Problem 6.3 (Solution) We follow the hint.

First, if $f : \mathbb{R}^{d\times n} \to \mathbb{R}$, $f = f(x_1, \dots, x_n)$, $x_1, \dots, x_n \in \mathbb{R}^d$, we see that

$$\begin{aligned}
\mathbb{E}^x f(B(t_1)), \dots, B(t_n))\\
&= \mathbb{E}\, f(B(t_1)) + x, \dots, B(t_n) + x)\\
&= \underbrace{\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}}_{n \text{ times}} f(y_1 + x, \dots, y_n + x)\ \mathbb{P}(B(t_1) \in dy_1, \dots, B(t_n) \in dy_n)
\end{aligned}$$

and the last expression is clearly measurable. This applies, in particular, to $f = \prod_{j=1}^n \mathbb{1}_{A_j}$ where $G := \bigcap_{j=1}^n \{B(t_j) \in A_j\}$, i.e. $\mathbb{E}^x \mathbb{1}_G$ is Borel measurable.

Set

$$\Gamma := \left\{\bigcap_{j=1}^n \{B(t_j) \in A_j\} \,:\, n \geqslant 0,\ 0 \leqslant t_1 < \cdots t_n,\ A_1, \dots A_n \in \mathcal{B}_b(\mathbb{R}^d)\right\}.$$

Let us see that Σ is a Dynkin system. Clearly, $\varnothing \in \Sigma$. If $A \in \Sigma$, then

$$x \mapsto \mathbb{E}^x \mathbb{1}_{A^c} = \mathbb{E}^x(1 - \mathbb{1}_A) = 1 - \mathbb{E}^x \mathbb{1}_A \in \mathcal{B}_b(\mathbb{R}^d) \implies A^c \in \Sigma.$$

Finally, if $(A_j)_{j\geqslant 1} \subset \Sigma$ are disjoint and $A := \bigcupdot_j A_j$ we get $\mathbb{1}_A = \sum_j \mathbb{1}_{A_j}$. Thus,

$$x \mapsto \mathbb{E}^x \mathbb{1}_A = \sum_j \mathbb{E}^x \mathbb{1}_{A_j} \in \mathcal{B}_b(\mathbb{R}^d).$$

This shows that Σ is a Dynkin System. Denote by $\delta(\cdot)$ the Dynkin system generated by the argument. Then

$$\Gamma \subset \Sigma \subset \mathcal{F}_\infty^B \implies \delta(\Gamma) \subset \delta(\Sigma) = \Sigma \subset \mathcal{F}_\infty^B.$$

But $\delta(\Gamma) = \sigma(\Gamma)$ since Γ is stable under finite intersections and $\sigma(\Gamma) = \mathcal{F}_\infty^B$. This proves, in particular, that $\Sigma = \mathcal{F}_\infty^B$.

Since we can approximate every bounded $\mathcal{F}_\infty^B$ measurable function Z by step functions with steps from $\mathcal{F}_\infty^B$, the claim follows.

Problem 6.4 (Solution) Following the hint we set $u_n(x) := (-n)\vee x\wedge n$. Then $u_n(x) \to u(x) := x$. Using (6.7) we see

$$\mathbb{E}\left[u_n(B_{t+\tau}) \,\middle|\, \mathcal{F}_{\tau+}\right](\omega) = \mathbb{E}^{B_\tau(\omega)} u_n(B_t).$$

Now take $t = 0$ to get

$$\mathbb{E}\left[u_n(B_\tau) \,\middle|\, \mathcal{F}_{\tau+}\right](\omega) = u_n(B_\tau)(\omega)$$

and we get

$$\lim_{n\to\infty} \mathbb{E}\left[u_n(B_\tau) \,\middle|\, \mathcal{F}_{\tau+}\right](\omega) = \lim_{n\to\infty} u_n(B_\tau)(\omega) = B_\tau(\omega).$$

Since the l.h.S. is $\mathcal{F}_{\tau+}$ measurable (as limit of such measurable functions!), the claim follows.

Problem 6.5 (Solution) By the reflection principle, Theorem 6.9,

$$\mathbb{P}\left(\sup_{s\leqslant t}|B_s| \geqslant x\right) \leqslant \mathbb{P}\left(\sup_{s\leqslant t} B_s \geqslant x\right) + \mathbb{P}\left(\inf_{s\leqslant t} B_s \leqslant -x\right) = \mathbb{P}(|B_t| \geqslant x) + \mathbb{P}(|B_t| \geqslant x).$$

Problem 6.6 (Solution) a) Since $B(\cdot) \sim -B(\cdot)$, we get

$$\tau_b = \inf\{s \geqslant 0 \,:\, B_s = b\} \sim \inf\{s \geqslant 0 \,:\, -B_s = b\} = \inf\{s \geqslant 0 \,:\, B_s = -b\} = \tau_{-b}.$$

b) Since $B(c^{-2}\,\cdot) \sim c^{-1}\,B(\cdot)$, we get

$$\begin{aligned}\tau_{cb} = \inf\{s \geqslant 0 \,:\, B_s = cb\} &= \inf\{s \geqslant 0 \,:\, c^{-1}\,B_s = b\}\\ &\sim \inf\{s \geqslant 0 \,:\, B_{s/c^2} = b\}\\ &= \inf\{rc^2 \geqslant 0 \,:\, B_r = b\}\\ &= c^2 \inf\{r \geqslant 0 \,:\, B_r = b\} = c^2\tau_b.\end{aligned}$$

c) We have

$$\tau_b - \tau_a = \inf\{s \geqslant 0 \,:\, B_{s+\tau_a} = b\} = \inf\{s \geqslant 0 \,:\, B_{s+\tau_a} - B_{\tau_a} = b - a\}$$

which shows that $\tau_b - \tau_a$ is independent of $\mathcal{F}_{\tau_a}$ by the strong Markov property of Brownian motion.

Now we find for all $s, t \geqslant 0$ and $c \in [0, a]$

$$\{\tau_c \leqslant s\} \cap \{\tau_a \leqslant t\} \overset{\tau_c\leqslant\tau_a}{=} \{\tau_c \leqslant s \wedge t\} \cap \{\tau_a \leqslant t\} \in \mathcal{F}_{t\wedge s} \cap \mathcal{F}_t \subset \mathcal{F}_t.$$

This shows that $\{\tau_c \leqslant s\} \in \mathcal{F}_{\tau_a}$, i.e. τ_c is $\mathcal{F}_{\tau_a}$ measurable. Since c is arbitrary, $\{\tau_c\}_{c\in[0,a]}$ is $\mathcal{F}_{\tau_a}$ measurable, and the claim follows.

Problem 6.7 (Solution) We begin with a simpler situation. As usual, we write τ_b for the first passage time of the level b: $\tau_b = \inf\{t \geqslant 0 \,:\, \sup_{s\leqslant t} B_s = b\}$ where $b > 0$. From Example 5.2 d) we know that $(M_t^\xi := \exp(\xi B_t - \frac{1}{2}t\xi^2))_{t\geqslant 0}$ is a martingale. By optional stopping we get that $(M^\xi_{t\wedge\tau_b})_{t\geqslant 0}$ is also a martingale and has, therefore, constant expectation. Thus, for $\xi > 0$ (and with $\mathbb{E} = \mathbb{E}^0$)

$$1 = \mathbb{E}\, M_0^\xi = \mathbb{E}\left(\exp(\xi B_{t\wedge\tau_b} - \tfrac12 (t\wedge\tau_b)\xi^2)\right)$$

Since the RV $\exp(\xi B_{t\wedge\tau_b})$ is bounded (mind: $\xi \geqslant 0$ and $B_{t\wedge\tau_b} \leqslant b$), we can let $t \to \infty$ and get

$$1 = \mathbb{E}\left(\exp(\xi B_{\tau_b} - \tfrac12 \tau_b\xi^2)\right) = \mathbb{E}\left(\exp(\xi b - \tfrac12 \tau_b\xi^2)\right)$$

or, if we take $\xi = \sqrt{2\lambda}$,

$$\mathbb{E}\, e^{-\lambda\tau_b} = e^{-\sqrt{2\lambda}b}.$$

As $B \sim -B$, $\tau_b \sim \tau_{-b}$, and the above calculation yields

$$\mathbb{E}\, e^{-\lambda\tau_b} = e^{-\sqrt{2\lambda}|b|} \qquad \forall b \in \mathbb{R}.$$

Now let us turn to the situation of the problem. Set $\tau = \tau^\circ_{(a,b)^c}$. Here, $B_{t\wedge\tau}$ is bounded (it is in the interval (a,b), and this makes things easier when it comes to optional stopping. As before, we get by stopping the martingale $(M^\xi_t)_{t\geqslant 0}$ that

$$e^{\xi x} = \lim_{t\to\infty} \mathbb{E}^x\left(\exp(\xi B_{t\wedge\tau} - \tfrac12 (t\wedge\tau)\xi^2)\right) = \mathbb{E}^x\left(\exp(\xi B_\tau - \tfrac12 \tau\xi^2)\right) \qquad \forall \xi$$

(and not, as before, for positive ξ! Mind also the starting point $x \neq 0$, but this does not change things dramatically.) by, e.g., dominated convergence. The problem is now that B_τ does not attain a particular value as it may be a or b. We get, therefore, for all $\xi \in \mathbb{R}$

$$\begin{aligned} e^{\xi x} &= \mathbb{E}^x\left(\exp(\xi B_\tau - \tfrac12\tau\xi^2)\mathbb{1}_{\{B_\tau = a\}}\right) + \mathbb{E}^x\left(\exp(\xi B_\tau - \tfrac12\tau\xi^2)\mathbb{1}_{\{B_\tau = b\}}\right) \\ &= \mathbb{E}^x\left(\exp(\xi a - \tfrac12\tau\xi^2)\mathbb{1}_{\{B_\tau = a\}}\right) + \mathbb{E}^x\left(\exp(\xi b - \tfrac12\tau\xi^2)\mathbb{1}_{\{B_\tau = b\}}\right) \end{aligned}$$

Now pick $\xi = \pm\sqrt{2\lambda}$. This yields 2 equations in two unknowns:

$$\begin{aligned} e^{\sqrt{2\lambda}\,x} &= e^{\sqrt{2\lambda}\,a}\,\mathbb{E}^x\left(e^{-\lambda\tau}\mathbb{1}_{\{B_\tau=a\}}\right) + e^{\sqrt{2\lambda}\,b}\,\mathbb{E}^x\left(e^{-\lambda\tau}\mathbb{1}_{\{B_\tau=b\}}\right) \\ e^{-\sqrt{2\lambda}\,x} &= e^{-\sqrt{2\lambda}\,a}\,\mathbb{E}^x\left(e^{-\lambda\tau}\mathbb{1}_{\{B_\tau=a\}}\right) + e^{-\sqrt{2\lambda}\,b}\,\mathbb{E}^x\left(e^{-\lambda\tau}\mathbb{1}_{\{B_\tau=b\}}\right) \end{aligned}$$

Solving this system of equations gives

$$\begin{aligned} e^{\sqrt{2\lambda}\,(x-a)} &= \mathbb{E}^x\left(e^{-\lambda\tau}\mathbb{1}_{\{B_\tau=a\}}\right) + e^{\sqrt{2\lambda}\,(b-a)}\,\mathbb{E}^x\left(e^{-\lambda\tau}\mathbb{1}_{\{B_\tau=b\}}\right) \\ e^{-\sqrt{2\lambda}\,(x-a)} &= \mathbb{E}^x\left(e^{-\lambda\tau}\mathbb{1}_{\{B_\tau=a\}}\right) + e^{-\sqrt{2\lambda}\,(b-a)}\,\mathbb{E}^x\left(e^{-\lambda\tau}\mathbb{1}_{\{B_\tau=b\}}\right) \end{aligned}$$

and so

$$\mathbb{E}^x\left(e^{-\lambda\tau}\mathbb{1}_{\{B_\tau=b\}}\right) = \frac{\sinh\left(\sqrt{2\lambda}\,(x-a)\right)}{\sinh\left(\sqrt{2\lambda}\,(b-a)\right)} \quad\text{and}\quad \mathbb{E}^x\left(e^{-\lambda\tau}\mathbb{1}_{\{B_\tau=a\}}\right) = \frac{\sinh\left(\sqrt{2\lambda}\,(b-x)\right)}{\sinh\left(\sqrt{2\lambda}\,(b-a)\right)}.$$

$\boxed{\text{This answers Problem b)}}$.

For the $\boxed{\text{solution of Problem a)}}$ we only have to add these two expressions:

$$\mathbb{E}\, e^{-\lambda\tau} = \mathbb{E}\left(e^{-\lambda\tau}\mathbb{1}_{\{B_\tau=a\}}\right) + \mathbb{E}\left(e^{-\lambda\tau}\mathbb{1}_{\{B_\tau=b\}}\right) = \frac{\sinh\left(\sqrt{2\lambda}\,(b-x)\right) + \sinh\left(\sqrt{2\lambda}\,(x-a)\right)}{\sinh\left(\sqrt{2\lambda}\,(b-a)\right)}.$$

Problem 6.8 (Solution) Solution 1 (direct calculation): Denote by $\tau = \tau_y = \inf\{s > 0 : B_s = y\}$ the first passage time of the level y. Then $B_\tau = y$ and we get for $y \geqslant x$

$$\begin{aligned}
\mathbb{P}(B_t \leqslant x,\ M_t \geqslant y) &= \mathbb{P}(B_t \leqslant x,\ \tau \leqslant t) \\
&= \mathbb{P}(B_{t\vee\tau} \leqslant x,\ \tau \leqslant t) \\
&= \mathbb{E}\Big(\mathbb{E}\left(\mathbb{1}_{\{B_{t\vee\tau}\leqslant x\}} \,\middle|\, \mathcal{F}_{\tau+}\right) \cdot \mathbb{1}_{\{\tau\leqslant t\}}\Big)
\end{aligned}$$

by the tower property and pull-out. Now we can use Theorem 6.11

$$\begin{aligned}
&= \int \mathbb{P}^{B_\tau(\omega)}(B_{t-\tau(\omega)} \leqslant x) \cdot \mathbb{1}_{\{\tau\leqslant t\}}(\omega)\ \mathbb{P}(d\omega) \\
&= \int \mathbb{P}^{y}(B_{t-\tau(\omega)} \leqslant x) \cdot \mathbb{1}_{\{\tau\leqslant t\}}(\omega)\ \mathbb{P}(d\omega) \\
&= \int \mathbb{P}(B_{t-\tau(\omega)} \leqslant x - y) \cdot \mathbb{1}_{\{\tau\leqslant t\}}(\omega)\ \mathbb{P}(d\omega) \\
&\overset{B\sim -B}{=} \int \mathbb{P}(B_{t-\tau(\omega)} \geqslant y - x) \cdot \mathbb{1}_{\{\tau\leqslant t\}}(\omega)\ \mathbb{P}(d\omega) \\
&= \int \mathbb{P}^{y}(B_{t-\tau(\omega)} \geqslant 2y - x) \cdot \mathbb{1}_{\{\tau\leqslant t\}}(\omega)\ \mathbb{P}(d\omega) \\
&= \int \mathbb{P}^{B_\tau(\omega)}(B_{t-\tau(\omega)} \geqslant 2y - x) \cdot \mathbb{1}_{\{\tau\leqslant t\}}(\omega)\ \mathbb{P}(d\omega) \\
&= \ldots = \mathbb{P}(B_t \geqslant 2y - x,\ M_t \geqslant y) \overset{y\geqslant x}{=} \mathbb{P}(B_t \geqslant 2y - x).
\end{aligned}$$

This means that

$$\mathbb{P}(B_t \leqslant x,\ M_t \geqslant y) = \mathbb{P}(B_t \geqslant 2y - x) = \int_{2y-x}^{\infty} (2\pi t)^{-1/2} e^{-z^2/(2t)}\,dz$$

and differentiating in x and y yields

$$\mathbb{P}(B_t \in dx,\ M_t \in dy) = \frac{2(2y-x)}{\sqrt{2\pi t^3}}\, e^{-(2y-x)^2/(2t)}\,dx\,dy.$$

Solution 2 (using Theorem 6.18): We have (with the notation of Theorem 6.18)

$$\mathbb{P}(M_t < y, B_t \in dx) = \lim_{a\to-\infty} \mathbb{P}(m_t > a, M_t < y, B_t \in dx) \overset{(6.19)}{=} \frac{dx}{\sqrt{2\pi t}}\left[e^{-\frac{x^2}{2t}} - e^{-\frac{(x-2y)^2}{2t}}\right]$$

and if we differentiate this expression in y we get

$$\mathbb{P}(B_t \in dx,\ M_t \in dy) = \frac{2(2y-x)}{\sqrt{2\pi t^3}}\, e^{-(2y-x)^2/(2t)}\,dx\,dy.$$

Problem 6.9 (Solution) This is the so-called *absorbed* or *killed Brownian motion*. The result is

$$\mathbb{P}^x(B_t \in dz,\ \tau_0 > t) = \big(g_t(x-z) - g_t(x+z)\big)\,dz = \frac{1}{\sqrt{2\pi t}}\Big(e^{-(x-z)^2/(2t)} - e^{-(x+z)^2/(2t)}\Big)\,dz,$$

for $x, z > 0$ or $x, z < 0$.

To see this result we assume that $x > 0$. Write $M_t = \sup_{s\leqslant t} B_s$ and $m_t = \inf_{s\leqslant t} B_s$ for the running maximum and minimum, respectively. Then we have for $A \subset [0,\infty)$

$$\mathbb{P}^x(B_t \in A,\ \ \tau_0 > t) = \mathbb{P}^x(B_t \in A,\ \ m_t > 0)$$

$$= \mathbb{P}^x(B_t \in A,\ \ x \geqslant m_t > 0)$$

(we start in $x > 0$, so the minimum is smaller!)

$$\begin{aligned}
&= \mathbb{P}^0(B_t \in A - x,\ \ 0 \geqslant m_t > -x)\\
&\overset{B \sim -B}{=} \mathbb{P}^0(-B_t \in A - x,\ \ 0 \geqslant -M_t > -x)\\
&= \mathbb{P}^0(B_t \in x - A,\ \ 0 \leqslant M_t < x)\\
&= \iint \mathbb{1}_A(x-a)\mathbb{1}_{[0,x)}(b)\ \mathbb{P}^0(B_t \in da,\ M_t \in db)
\end{aligned}$$

Now we use the result of Problem 6.8:

$$\mathbb{P}^0(B_t \in da,\ M_t \in db) = \frac{2(2b-a)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2b-a)^2}{2t}\right) da\, db$$

and we get

$$\begin{aligned}
\mathbb{P}^x(B_t \in A,\ \tau_0 > t) &= \int \mathbb{1}_A(x-a)\left[\int_0^x \frac{2(2b-a)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2b-a)^2}{2t}\right) db\right] da\\
&= \int \mathbb{1}_A(x-a)\frac{t}{\sqrt{2\pi t^3}}\left[\int_0^x \frac{2\cdot 2\cdot(2b-a)}{2t} \exp\left(-\frac{(2b-a)^2}{2t}\right) db\right] da\\
&= \int \mathbb{1}_A(x-a)\frac{1}{\sqrt{2\pi t}}\left[\int_0^x \frac{2\cdot(2b-a)}{t} \exp\left(-\frac{(2b-a)^2}{2t}\right) db\right] da\\
&= \frac{1}{\sqrt{2\pi t}} \int \mathbb{1}_A(x-a)\left[-\exp\left(-\frac{(2b-a)^2}{2t}\right)\right]_{b=0}^{x} da\\
&= \frac{1}{\sqrt{2\pi t}} \int \mathbb{1}_A(x-a)\left\{\exp\left(-\frac{a^2}{2t}\right) - \exp\left(-\frac{(2x-a)^2}{2t}\right)\right\} da\\
&= \frac{1}{\sqrt{2\pi t}} \int \mathbb{1}_A(z)\left\{\exp\left(-\frac{(x-z)^2}{2t}\right) - \exp\left(-\frac{(x+z)^2}{2t}\right)\right\} da.
\end{aligned}$$

The calculation for $x < 0$ is similar (actually easier): Let $A \subset (-\infty, 0]$

$$\begin{aligned}
\mathbb{P}^x(B_t \in A,\ \tau_0 > 0) &= \mathbb{P}^x(B_t \in A,\ -x \leqslant M_t < 0)\\
&= \mathbb{P}^0(B_t \in A - x,\ \ 0 \leqslant M_t < -x)\\
&= \iint \mathbb{1}_A(a+x)\mathbb{1}_{[0,-x)}(b)\, \frac{2(2b-a)}{\sqrt{2\pi t^2}}\, \exp\left(-\frac{(2b-a)^2}{2t}\right) db\, da\\
&= \int \mathbb{1}_A(a+x)\frac{t}{\sqrt{2\pi t^3}} \int_0^{-x} \frac{2\cdot(2b-a)}{t}\, \exp\left(-\frac{(2b-a)^2}{2t}\right) db\, da\\
&= \frac{1}{\sqrt{2\pi t}} \int \mathbb{1}_A(a+x)\left[-\exp\left(-\frac{(2b-a)^2}{2t}\right)\right]_{b=0}^{-x} da\\
&= \frac{1}{\sqrt{2\pi t}} \int \mathbb{1}_A(a+x)\left\{\exp\left(-\frac{a^2}{2t}\right) - \exp\left(-\frac{(2x+a)^2}{2t}-\right)\right\} da\\
&= \frac{1}{\sqrt{2\pi t}} \int \mathbb{1}_A(y)\left\{\exp\left(-\frac{(y-x)^2}{2t}\right) - \exp\left(-\frac{(x+y)^2}{2t}-\right)\right\} da.
\end{aligned}$$

Problem 6.10 (Solution) For a compact set $K \subset \mathbb{R}^d$ the set $U_n := K + \mathbb{B}(0, 1/n) := \{x + y : x \in K,\ |y| < 1/n\}$ is open.

$$\phi_n(x) := d(x, U_n^c)/(d(x,K) + d(x, U_n^c)).$$

Since for $d(x,z) := |x-z|$ and all $x,z \in \mathbb{R}^d$

$$d(x,A) \leqslant d(x,z) + d(z,A) \implies |d(x,A) - d(z,A)| \leqslant d(x,z),$$

we see that $\phi_n(x)$ is continuous. Obviously, $\mathbb{1}_{U_n}(x) \geqslant \phi_n(x) \geqslant \phi_{n+1} \geqslant \mathbb{1}_K$, and $\mathbb{1}_K = \inf_n \phi_n$ follows.

Problem 6.11 (Solution) Recall that $\mathbb{P} = \mathbb{P}^0$. We have for all $a \geqslant t \geqslant 0$

$$\begin{aligned}
\mathbb{P}(\widetilde{\xi}_t > a) &= \mathbb{P}\left(\inf\left\{s \geqslant t : B_s = 0\right\} > a\right)\\
&= \mathbb{P}\left(\inf\left\{h \geqslant 0 : B_{t+h} = 0\right\} + t > a\right)\\
&= \mathbb{E}\left[\mathbb{P}^{B_t}\left(\inf\left\{h \geqslant 0 : B_h = 0\right\} > a-t\right)\right]\\
&= \mathbb{E}\left[\mathbb{P}^0\left(\inf\left\{h \geqslant 0 : B_h + x = 0\right\} > a-t\right)\Big|_{x=B_t}\right]\\
&= \mathbb{E}\left[\mathbb{P}\left(\inf\left\{h \geqslant 0 : B_h = -x\right\} > a-t\right)\Big|_{x=B_t}\right]\\
&= \mathbb{E}\left[\mathbb{P}\left(\tau_{-x} > a-t\right)\Big|_{x=B_t}\right]\\
&\overset{B\sim -B}{=} \mathbb{E}\left[\mathbb{P}\left(\tau_{B_t} > a-t\right)\right]\\
&\overset{(6.13)}{=} \mathbb{E}\left[\int_{a-t}^{\infty} \frac{|B_t|}{\sqrt{2\pi s^3}}\, e^{-B_t^2/(2s)}\, ds\right]\\
&= \int_{a-t}^{\infty} \mathbb{E}\left[\frac{|B_t|}{\sqrt{2\pi s^3}}\, e^{-B_t^2/(2s)}\right] ds.
\end{aligned}$$

Thus, differentiating with respect to a and using Brownian scaling yields

$$\begin{aligned}
\mathbb{P}(\widetilde{\xi}_t \in da) &= \mathbb{E}\left[\frac{|B_t|}{\sqrt{2\pi(a-t)^3}}\, \exp\left(-\frac{B_t^2}{2(a-t)}\right)\right]\\
&= \frac{1}{(a-t)\sqrt{\pi}}\, \mathbb{E}\left[\frac{\sqrt{t}}{\sqrt{a-t}}\, \frac{|B_1|}{\sqrt{2}}\, \exp\left(-\frac{1}{2} B_1^2 \frac{t}{a-t}\right)\right]\\
&= \frac{1}{(a-t)\sqrt{\pi}}\, \mathbb{E}\left[|c\,B_1|\, \exp\left(-(c\,B_1)^2\right)\right]\\
&= \frac{1}{(a-t)\sqrt{\pi}}\, \mathbb{E}\left[|B_{c^2}|\, \exp\left(-B_{c^2}^2\right)\right]
\end{aligned}$$

where $c^2 = \frac{1}{2}\,\frac{t}{a-t}$.

Now let us calculate for $s = c^2$

$$\begin{aligned}
\mathbb{E}\left[|B_s|\, e^{-B_s^2}\right] &= (2\pi s)^{-1/2} \int_{-\infty}^{\infty} |x|\, e^{-x^2}\, e^{-x^2/(2s)}\, dx\\
&= (2\pi s)^{-1/2}\, 2 \int_0^{\infty} x\, e^{-x^2(1+(2s)^{-1})}\, dx\\
&= (2\pi s)^{-1/2} \frac{1}{(1+(2s)^{-1})} \int_0^{\infty} 2(1+(2s)^{-1}) x\, e^{-x^2(1+(2s)^{-1})}\, dx\\
&= \frac{1}{\sqrt{2\pi s}}\, \frac{2s}{2s+1} \left[e^{-x^2(1+(2s)^{-1})}\right]_{x=0}^{\infty}\\
&= \frac{1}{\sqrt{2\pi s}}\, \frac{2s}{2s+1}.
\end{aligned}$$

Let $(B_t)_{t\geqslant 0}$ be a BM[1]. Find the distribution of $\widetilde{\xi}_t := \inf\{s \geqslant t : B_s = 0\}$. This gives

$$\begin{aligned}\mathbb{P}(\widetilde{\xi}_t \in da) &= \frac{1}{(a-t)\sqrt{\pi}}\,\frac{1}{\sqrt{2\pi}\,c}\,\frac{2c^2}{2c^2+1}\\ &= \frac{1}{(a-t)\pi}\,\frac{\sqrt{a-t}}{\sqrt{t}}\,\frac{t}{(a-t)a/(a-t)}\\ &= \frac{1}{a\pi}\sqrt{\frac{t}{a-t}}.\end{aligned}$$

Problem 6.12 (Solution) a) We have

$$\mathbb{P}\left(B_t = 0 \quad \text{for some} \quad t \in (u,v)\right) = 1 - \mathbb{P}\left(B_t \neq 0 \quad \text{for all} \quad t \in (u,v)\right).$$

But the complementary probability is known from Theorem 6.19.

$$\mathbb{P}\left(B_t \neq 0 \quad \text{for all} \quad t \in (u,v)\right) = \frac{2}{\pi}\arcsin\sqrt{\frac{u}{v}}$$

and so

$$\mathbb{P}\left(B_t = 0 \quad \text{for some} \quad t \in (u,v)\right) = 1 - \frac{2}{\pi}\arcsin\sqrt{\frac{u}{v}}.$$

b) Since $(u,v) \subset (u,w)$ we find with the classical conditional probability that

$$\begin{aligned}&\mathbb{P}\left(B_t \neq 0\ \forall t \in (u,w) \,\middle|\, B_t \neq 0\ \forall t \in (u,v)\right)\\ &\qquad= \frac{\mathbb{P}\left(\{B_t \neq 0\ \forall t \in (u,w)\} \cap \{B_t \neq 0\ \forall t \in (u,v)\}\right)}{\mathbb{P}\left(B_t \neq 0\ \forall t \in (u,v)\right)}\\ &\qquad= \frac{\mathbb{P}\left(B_t \neq 0\ \forall t \in (u,w)\right)}{\mathbb{P}\left(B_t \neq 0\ \forall t \in (u,v)\right)}\\ &\qquad\overset{\text{a)}}{=} \frac{\arcsin\sqrt{\frac{u}{w}}}{\arcsin\sqrt{\frac{u}{v}}}\end{aligned}$$

c) We have

$$\begin{aligned}&\mathbb{P}\left(B_t \neq 0\ \forall t \in (0,w) \,\middle|\, B_t \neq 0\ \forall t \in (0,v)\right)\\ &\qquad= \lim_{u\to 0}\mathbb{P}\left(B_t \neq 0\ \forall t \in (u,w) \,\middle|\, B_t \neq 0\ \forall t \in (u,v)\right)\\ &\qquad\overset{\text{b)}}{=} \lim_{u\to 0}\frac{\arcsin\sqrt{\frac{u}{w}}}{\arcsin\sqrt{\frac{u}{v}}}\\ &\qquad\underset{\text{l'Hôpital}}{\overset{\text{a)}}{=}} \lim_{u\to 0}\frac{\sqrt{v}\sqrt{v-u}}{\sqrt{w}\sqrt{w-u}}\\ &\qquad= \frac{\sqrt{v}}{\sqrt{w}}.\end{aligned}$$

Problem 6.13 (Solution) We have seen in Problem 6.1 that $M - B$ is a Markov process with the same law as $|B|$. This entails immediately that $\xi \sim \eta$.

Attention: this problem shows that it is not enough to have only $M_t - B_t \sim |B_t|$ for all $t \geqslant 0$, we do need that the finite-dimensional distributions coincide. The Markov property guarantees just this once the one-dimensional distributions coincide!

7 Brownian Motion and Transition Semigroups

Problem 7.1 (Solution) Banach space: It is obvious that $\mathcal{C}_\infty(\mathbb{R}^d)$ is a linear space. Let us show that it is closed. By definition, $u \in \mathcal{C}_\infty(\mathbb{R}^d)$ if

$$\forall \epsilon > 0 \quad \exists R > 0 \quad \forall |x| > R : |u(x)| < \epsilon. \tag{*}$$

Let $(u_n)_n \subset \mathcal{C}_\infty(\mathbb{R}^d)$ be a Cauchy sequence for the uniform convergence. It is clear that the uniform limit $u = \lim_n u_n$ is again continuous. Fix ϵ and pick R as in (*). Then we get

$$|u(x)| \leqslant |u_n(x) - u(x)| + |u_n(x)| \leqslant \|u_n - u\|_\infty + |u_n(x)|.$$

By uniform convergence, there is some $n(\epsilon)$ such that

$$|u(x)| \leqslant \epsilon + |u_{n(\epsilon)}(x)| \quad \text{for all} \quad x \in \mathbb{R}^d.$$

Since $u_{n(\epsilon)} \in \mathcal{C}_\infty$, we find with (*) some $R = R(n(\epsilon), \epsilon) = R(\epsilon)$ such that

$$|u(x)| \leqslant \epsilon + |u_{n(\epsilon)}(x)|. \leqslant \epsilon + \epsilon \quad \forall |x| > R(\epsilon).$$

Density: Fix an ϵ and pick $R > 0$ as in (*), and pick a cut-off function $\chi = \chi_R \in \mathcal{C}(\mathbb{R}^d)$ such that

$$\mathbb{1}_{\overline{\mathbb{B}}(0,R)} \leqslant \chi_R \leqslant \mathbb{1}_{\mathbb{B}(0,2R)}.$$

Clearly, $\operatorname{supp} \chi_R$ is compact, $\chi_R \uparrow 1$, $\chi_R u \in \mathcal{C}_c(\mathbb{R}^d)$ and

$$\sup_x |u(x) - \chi_R(x)u(x)| = \sup_{|x|>R} |\chi_R(x)u(x)| \leqslant \sup_{|x|>R} |u(x)| < \epsilon.$$

This shows that $\mathcal{C}_c(\mathbb{R}^d)$ is dense in $\mathcal{C}_\infty(\mathbb{R}^d)$.

Problem 7.2 (Solution) Fix $(t, y, v) \in [0, \infty) \times \mathbb{R}^d \times \mathcal{C}_\infty(\mathbb{R}^d)$, $\epsilon > 0$, and take any $(s, x, u) \in [0, \infty) \times \mathbb{R}^d \times \mathcal{C}_\infty(\mathbb{R}^d)$. Then we find using the triangle inequality

$$\begin{aligned}
|P_s u(x) - P_t v(y)| &\leqslant |P_s u(x) - P_s v(x)| + |P_s v(x) - P_t v(x)| + |P_t v(x) - P_t v(y)| \\
&\leqslant \sup_x |P_s u(x) - P_s v(x)| + \sup_x |P_s v(x) - P_s P_{t-s} v(x)| + |P_t v(x) - P_t v(y)| \\
&= \|P_s(u - v)\|_\infty + \|P_s(v - P_{t-s}v)\|_\infty + |P_t v(x) - P_t v(y)| \\
&\leqslant \|u - v\|_\infty + \|v - P_{t-s}v\|_\infty + |P_t v(x) - P_t v(y)|
\end{aligned}$$

where we used the contraction property of P_s.

- Since $y \mapsto P_t v(y)$ is continuous, there is some $\delta_1 = \delta_1(t, y, v, \epsilon)$ such that $|x - y| < \delta \implies |P_t v(x) - P_t v(y)| < \epsilon$.
- Using the strong continuity of the semigroup (Proposition 7.3 f) there is some $\delta_2 = \delta_2(t, v, \epsilon)$ such that $|t - s| < \delta_2 \implies \|P_{t-s} v - v\|_\infty \leqslant \epsilon$.

. This proves that for $\delta := \min\{\epsilon, \delta_1, \delta_2\}$

$$|s - t| + |x - y| + \|u - v\|_\infty \leqslant \delta \implies |P_s u(x) - P_t v(y)| \leqslant 3\epsilon.$$

Problem 7.3 (Solution) By the tower property we find

$$\begin{aligned}
\mathbb{E}^x(f(X_t)g(X_{t+s})) &\overset{\text{tower}}{\underset{\text{property}}{=}} \mathbb{E}^x\left(\mathbb{E}^x\left(f(X_t)g(X_{t+s}) \,\middle|\, \mathcal{F}_t\right)\right) \\
&\overset{\text{pull}}{\underset{\text{out}}{=}} \mathbb{E}^x\left(f(X_t)\,\mathbb{E}^x\left(g(X_{t+s}) \,\middle|\, \mathcal{F}_t\right)\right) \\
&\overset{\text{Markov}}{\underset{\text{property}}{=}} \mathbb{E}^x\left(f(X_t)\,\mathbb{E}^{X_t}\left(g(X_s)\right)\right) \\
&= \mathbb{E}^x(f(X_t)h(X_t))
\end{aligned}$$

where, for every s,

$$h(y) = \mathbb{E}^y g(X_s) \quad \text{is again in} \quad \mathcal{C}_\infty.$$

Thus, $\mathbb{E}^x f(X_t)g(X_{t+s}) = \mathbb{E}^x \phi(X_t)$ and $\phi(y) = f(y)h(y)$ is in $\mathcal{C}_\infty$. This shows that $x \mapsto \mathbb{E}^x(f(X_t)g(X_{t+s}))$ is in $\mathcal{C}_\infty$.

Using semigroups we can write the above calculation in the following form:

$$\mathbb{E}^x(f(X_t)g(X_{t+s})) = \mathbb{E}^x(f(X_t)P_s g(X_t)) = P_t(f P_s g)(x)$$

i.e. $h = P_s$ and $\phi = f \cdot P_s g$, and since P_t preserves $\mathcal{C}_\infty$, the claim follows.

Problem 7.4 (Solution) Set $u(t,z) := P_t u(z) = p_t \star u(z) = (2\pi t)^{d/2} \int_{\mathbb{R}^d} u(y) e^{|z-y|^2/2t}\, dy$.

<u>$u(t,\cdot)$ is in $\mathcal{C}^\infty$ for $t > 0$:</u> Note that the Gauss kernel

$$p_t(z - y) = (2\pi t)^{-d/2} e^{-|z-y|^2/2t}, \quad t > 0$$

can be arbitrarily often differentiated in z and

$$\partial_z^k p_t(z - y) = Q_k(z, y, t) p_t(z - y)$$

where the function $Q_k(z, y, t)$ grows at most polynomially in z and y. Since $p_t(z - y)$ decays exponentially, we see — as in the proof of Proposition 7.3 g) — that for each z

$$\begin{aligned}
&|\partial_z^k p_t(z - y)| \\
&\leqslant \sup_{|y| \leqslant 2R} \left|Q_k(z, y, t)\right| \mathbb{1}_{\mathbb{B}(0,2R)}(y) + \sup_{|y| \geqslant 2R} \left|Q_k(z, y, t) e^{-|y|^2/(16t)}\right| e^{-|y|^2/(16t)}\, \mathbb{1}_{\mathbb{B}^c(0,2R)}(y).
\end{aligned}$$

This inequality holds uniformly in a small neighbourhood of z, i.e. we can use the differentiation lemma from measure and integration to conclude that $\partial^k P_t u \in \mathcal{C}_b$.

$x \mapsto \partial_t u(t,x)$ is in $\mathcal{C}^\infty$ for $t > 0$: This follows from the first part and the fact that

$$\begin{aligned}\partial_t p_t(z-y) &= -\frac{d}{2}(2\pi t)^{-d/2-1}e^{-|z-y|^2/2t} + (2\pi t)^{-d/2}e^{-|z-y|^2/2t}\,\frac{|z-y|^2}{2t^2}\\ &= \frac{1}{2}\left(\frac{|z-y|^2}{t^2} - \frac{d}{t}\right)p_t(z-y).\end{aligned}$$

Again with the domination argument of the first part we see that $\partial_t\partial_x^k u(t,x)$ is continuous on $(0,\infty)\times\mathbb{R}^d$.

Problem 7.5 (Solution) (a) Note that $|u_n| \leqslant |u| \in L^p$. Since $|u_n - u|^p \leqslant (|u_n|+|u|)^p \leqslant (|u|+|u|)^p = 2^p|u|^p \in L^1$ and since $|u_n(x) - u(x)| \to 0$ for every x as $n \to \infty$, the claim follows by dominated convergence.

(b) Let $u \in L^p$ and $m < n$. We have

$$\|P_t u_n - P_t u_m\|_{L^p} = \|p_t \star (u_n - u_m)\|_{L^p} \overset{\text{Young}}{\leqslant} \|p_t\|_{L^1}\|u_n - u_m\|_{L^p} = \|u_n - u_m\|_{L^p}.$$

Since $(u_n)_n$ is an L^p Cauchy sequence (it converges in L^p towards $u \in L^p$), so is $(P_t u_n)_n$, and therefore $\tilde{P}_t u := \lim_n P_t u_n$ exists in L^p.

If v_n is any other sequence in L^p with limit u, the above argument shows that $\lim_n P_t v_n$ also exists. 'Mixing' the sequences $(w_n) := (u_1, v_1, u_2, v_2, u_3, v_3, \dots)$ produces yet another convergent sequence with limit u, and we conclude that

$$\lim_n P_t u_n = \lim_n P_t w_n = \lim_n P_t v_n,$$

i.e. $\tilde{P}_t$ is well-defined.

(c) Any $u \in L^p$ with $0 \leqslant u \leqslant 1$ has a representative $u \in \mathcal{B}_b$. And then the claim follows since P_t is sub-Markovian.

(d) Recall that $y \mapsto \|u(\cdot + y) - u\|_{L^p}$ is for $u \in L^p(dx)$ a continuous function. By Fubini's theorem and the Hölder inequality

$$\begin{aligned}\|P_t u - u\|_{L^p}^p &= \int |\mathbb{E}\, u(x+B_t) - u(x)|^p\,dx\\ &\leqslant \mathbb{E}\left(\int |u(x+B_t) - u(x)|^p\,dx\right)\\ &= \mathbb{E}\left(\|u(\cdot + B_t) - u\|_{L^p}^p\right).\end{aligned}$$

The integrand is bounded by $2^p\|u\|_{L^p}^p$, and continuous as a function of t; therefore we can use the dominated convergence theorem to conclude that $\lim_{t\to 0}\|P_t u - u\|_{L^p} = 0$.

Problem 7.6 (Solution) Let $u \in \mathcal{C}_b$. Then we have, by definition

$$\begin{aligned}T_{t+s}u(x) &= \int_{\mathbb{R}^d} u(z)\,p_{t+s}(x,dz)\\ T_t(T_s u)(x) &= \int_{\mathbb{R}^d} T_s u(y)\,p_t(x,dy)\\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} u(z)\,p_s(y,dz)\,p_t(x,dy)\end{aligned}$$

$$= \int_{\mathbb{R}^d} u(z) \int_{\mathbb{R}^d} p_s(y, dz)\, p_t(x, dy)$$

By the semigroup property, $T_{t+s} = T_t T_s$, and we see that

$$p_{t+s}(x, dz) = \int_{\mathbb{R}^d} p_s(y, dz)\, p_t(x, dy).$$

If we pick $u = \mathbb{1}_C$, this formal equality becomes

$$p_{t+s}(x, C) = \int_{\mathbb{R}^d} p_s(y, C)\, p_t(x, dy).$$

Problem 7.7 (Solution) Using $T_t \mathbb{1}_C(x) = p_t(x, C) = \int \mathbb{1}_C(y)\, p_t(x, dy)$ we get

$$\begin{aligned}
&p^x_{t_1,\dots,t_n}(C_1 \times \ldots \times C_n) \\
&= T_{t_1}\Big(\mathbb{1}_{C_1}\Big[T_{t_2-t_1}\mathbb{1}_{C_2}\Big\{\cdots T_{t_{n-1}-t_{n-2}} \int \mathbb{1}_{C_n}(x_n)\, p_{t_n-t_{n-1}}(\cdot, dx_n)\cdots\Big\}\Big]\Big)(x) \\
&= T_{t_1}\Big(\mathbb{1}_{C_1}\Big[T_{t_2-t_1}\mathbb{1}_{C_2}\Big\{\cdots \int \mathbb{1}_{C_{n-1}}(x_{n-1}) \int \mathbb{1}_{C_n}(x_n)\, p_{t_n-t_{n-1}}(x_{n-1}, dx_n)\times \\
&\qquad\qquad \times p_{t_{n-1}-t_{n-2}}(\cdot, dx_{n-1})\cdots\Big\}\Big]\Big)(x) \\
&\ \vdots \\
&= \underbrace{\int \ldots \int}_{n \text{ integrals}} \mathbb{1}_{C_1}(x_1)\mathbb{1}_{C_2}(x_2)\cdots\mathbb{1}_{C_n}(x_n) p_{t_n-t_{n-1}}(x_{n-1}, dx_n) p_{t_{n-1}-t_{n-2}}(x_{n-2}, dx_{n-1})\times \\
&\qquad\qquad \cdots \times p_{t_2-t_1}(x_2, dx_2) p_{t_1}(x, dx_1) \\
&= \underbrace{\int \ldots \int}_{n \text{ integrals}} \mathbb{1}_{C_1\times\dots\times C_n}(x_1, \ldots, x_n) \prod_{j=1}^n p_{t_j-t_{j-1}}(x_{j-1}, dx_j)
\end{aligned}$$

(we set $t_0 := 0$ and $x_0 := x$).

This shows that $p^x_{t_1,\dots,t_n}(C_1 \times \ldots \times C_n)$ is the restriction of

$$p^x_{t_1,\dots,t_n}(\Gamma) = \underbrace{\int \ldots \int}_{n \text{ integrals}} \mathbb{1}_\Gamma(x_1, \ldots, x_n) \prod_{j=1}^n p_{t_j-t_{j-1}}(x_{j-1}, dx_j), \quad \Gamma \in \mathcal{B}(\mathbb{R}^{d\cdot n})$$

and the right-hand side clearly defines a probability measure. By the uniqueness theorem for measures, each measure is uniquely defined by its values on the rectangles, so we are done.

Problem 7.8 (Solution) (a) Let $x, y \in \mathbb{R}^d$ and $a \in A$. Then

$$\inf_{\alpha\in A} |x - \alpha| \leqslant |x - a| \leqslant |x - y| + |a - y|$$

Since this holds for all $a \in A$, we get

$$\inf_{\alpha\in A} |x - \alpha| \leqslant |x - y| + \inf_{a\in A} |a - y|$$

and, since x, y play symmetric roles,

$$|d(x, A) - d(y, A)| = \Big|\inf_{\alpha\in A} |x - \alpha| - \inf_{a\in A} |a - y|\Big| \leqslant |x - y|.$$

(b) By definition, $U_n = K + \mathbb{B}(0, 1/n)$ and $u_n(x) := \frac{d(x,U_n^c)}{d(x,K)+d(x,U_n^c)}$. Being a combination of continuous functions, see Part (a), u_n is clearly continuous. Moreover,

$$u_n|_K \equiv 1 \quad \text{and} \quad u_n|_{U_n^c} \equiv 0.$$

This shows that $\mathbb{1}_K \leqslant u_n \leqslant \mathbb{1}_{U_n^c} \xrightarrow{n\to\infty} \mathbb{1}_K$.

Picture: u_n is piecewise linear.

(c) Assume, without loss of generality, that $\operatorname{supp}\chi_n \subset \mathbb{B}(0, 1/n^2)$. Since $0 \leqslant u_n \leqslant 1$, we find

$$\chi_n \star u_n(x) = \int \chi_n(x-y)u_n(y)\,dy \leqslant \int \chi_n(x-y)\,dy = 1 \qquad \forall x.$$

Now we observe that for $\gamma \in (0,1)$

$$u_n(y) = \frac{d(y,U_n^c)}{d(y,K)+d(y,U_n^c)} \geqslant \frac{(1-\gamma)/n}{1/n} = 1-\gamma. \qquad \forall y \in K + \mathbb{B}(0,\gamma/n)$$

(Essentially this means that u_n is 'linear' for $x \in U_n \smallsetminus K$!). Thus, if $\gamma > 1/n$,

$$\begin{aligned}
\chi_n \star u_n(x) &= \int \chi_n(x-y)u_n(y)\,dy \\
&\geqslant (1-\gamma)\int \chi_n(x-y)\mathbb{1}_{K+\mathbb{B}(0,\gamma/n)}(y)\,dy \\
&= (1-\gamma)\int \chi_n(x-y)\mathbb{1}_{\mathbb{B}(0,1/n^2)}(x-y)\mathbb{1}_{K+\mathbb{B}(0,\gamma/n)}(y)\,dy \\
&= (1-\gamma)\int \chi_n(x-y)\mathbb{1}_{x+\mathbb{B}(0,1/n^2)}(y)\mathbb{1}_{K+\mathbb{B}(0,\gamma/n)}(y)\,dy \\
&\geqslant (1-\gamma)\int \chi_n(x-y)\mathbb{1}_{x+\mathbb{B}(0,1/n^2)}(y)\,dy \\
&= 1-\gamma \qquad \forall x \in K.
\end{aligned}$$

This shows that

$$1-\gamma \leqslant \liminf_n \chi_n \star u_n(x) \leqslant \limsup_n \chi_n \star u_n(x) \leqslant 1 \qquad \forall x \in K,$$

hence,

$$\lim_{n\to\infty} \chi_n \star u_n(x) = x \quad \text{for all} \quad x \in K.$$

On the other hand, if $x \in K^c$, there is some $n \geqslant 1$ such that $d(x,K) > \frac{1}{n} + \frac{1}{n^2}$. Since

$$\frac{1}{n} + \frac{1}{n^2} < d(x,K) \leqslant d(x,y) + d(y,K) \implies d(x,y) > \frac{1}{n^2} \quad \text{or} \quad d(y,K) > \frac{1}{n},$$

and so, using that $\operatorname{supp}\chi_n \subset \mathbb{B}(0,1/n^2)$ and $\operatorname{supp} u_n \subset K + \mathbb{B}(0,1/n)$,

$$\chi_n \star u_n(x) = \int \chi_n(x-y)u_n(y)\,dy = 0 \quad \forall x : d(x,K) > \frac{1}{n} + \frac{1}{n^2}.$$

It follows that $\lim_n \chi_n \star u_n(x) = 0$ for $x \in K^c$.

<u>Remark 1:</u> If we are just interested in a smooth function approximating $\mathbb{1}_K$ we could use $v_n := \chi_n \star \mathbb{1}_{K+\operatorname{supp} u_n}$ where $(\chi_n)_n$ is any sequence of type δ. Indeed, as before,

$$\chi_n \star \mathbb{1}_{K+\operatorname{supp} u_n}(x) = \int \chi_n(x-y)\mathbb{1}_{K+\operatorname{supp} u_n}(y)\,dy \leqslant \int \chi_n(x-y)\,dy = 1 \qquad \forall x.$$

For $x \in K$ we find

$$\begin{aligned}\chi_n \star \mathbb{1}_{K+\operatorname{supp} u_n}(x) &= \int \chi_n(x-y)\mathbb{1}_{K+\operatorname{supp} u_n}(y)\,dy\\ &= \int \chi_n(y)\mathbb{1}_{K+\operatorname{supp} u_n}(x-y)\,dy\\ &= \int \chi_n(y)\,dy\\ &= 1 \qquad \forall x \in K.\end{aligned}$$

As before we get $\chi_n \star \mathbb{1}_{K+\operatorname{supp} u_n}(x) = 0$ if $d(x,K) > 2/n$.

Thus, $\lim_n \chi_n \star \mathbb{1}_{K+\operatorname{supp} u_n}(x) = 0$ if $x \in K^c$.

Remark 2: The naive approach $\chi_n \star \mathbb{1}_K$ will, in general, not lead to a (pointwise everywhere) approximation of $\mathbb{1}_K$: consider $K = \{0\}$, then $\chi_n \star \mathbb{1}_K \equiv 0$. In fact, since $\mathbb{1}_K \in L^1$ we get $\chi_n \star \mathbb{1}_K \to \mathbb{1}_K$ in L^1 hence, for a subsequence, a.e. ...

Problem 7.9 (Solution) (a) Existence, contractivity: Let us, first of all, check that the series converges. Denote by $\|A\|$ any matrix norm in $\mathbb{R}^d$. Then we see

$$\|P_t\| = \left\|\sum_{j=0}^\infty \frac{(tA)^j}{j!}\right\| \leqslant \sum_{j=0}^\infty \frac{t^j\,\|A^j\|}{j!} \leqslant \sum_{j=0}^\infty \frac{t^j\,\|A\|^j}{j!} = e^{t\|A\|}.$$

This shows that, in general, P_t is not a contraction. We can make it into a contraction by setting $Q_t := e^{-t\|A\|}P_t$. It is clear that Q_t is again a semigroup, if P_t is a semigroup.

Semigroup property: This is shown using as for the one-dimensional exponential series. Indeed,

$$\begin{aligned}e^{(t+s)A} &= \sum_{k=0}^\infty \frac{(t+s)^k A^k}{k!}\\ &= \sum_{k=0}^\infty \sum_{j=0}^k \frac{1}{k!}\binom{k}{j} t^j s^{k-j} A^k\\ &= \sum_{k=0}^\infty \sum_{j=0}^k \frac{t^j A^j}{j!}\,\frac{s^{k-j}A^{k-j}}{(k-j)!}\\ &= \sum_{j=0}^\infty \frac{t^j A^j}{j!} \sum_{k=j}^\infty \frac{s^{k-j}A^{k-j}}{(k-j)!}\\ &= \sum_{j=0}^\infty \frac{t^j A^j}{j!} \sum_{l=0}^\infty \frac{s^l A^l}{l!}\\ &= e^{tA}e^{sA}.\end{aligned}$$

Strong continuity: We have

$$\left\|e^{tA} - \mathrm{id}\right\| = \left\|\sum_{j=1}^\infty \frac{t^j A^j}{j!}\right\| = t\left\|\sum_{j=1}^\infty \frac{t^{j-1}A^j}{j!}\right\|$$

and, as in the first calculation, we see that the series converges absolutely. Letting $t \to 0$ shows strong continuity, even continuity in the operator norm.

(Strictly speaking, strong continuity means that for each vector $v \in \mathbb{R}^d$

$$\lim_{t\to 0} |e^{tA}v - v| = 0.$$

Since

$$|e^{tA}v - v| \leqslant \|e^{tA} - \mathrm{id}\| \cdot |v|$$

strong continuity is implied by uniform continuity. One can show that the generator of a norm-continuous semigroup is already a bounded operator, see e.g. Pazy.)

(b) Let $s, t > 0$. Then

$$e^{tA} - e^{sA} = \sum_{j=0}^{\infty} \left(\frac{t^j A^j}{j!} - \frac{s^j A^j}{j!} \right) = \sum_{j=1}^{\infty} \frac{(t^j - s^j) A^j}{j!}$$

Since the sum converges absolutely, we get

$$\frac{e^{tA} - e^{sA}}{t - s} = \sum_{j=1}^{\infty} \frac{(t^j - s^j)}{t - s} \frac{A^j}{j!} \xrightarrow{s\to t} \sum_{j=1}^{\infty} j t^{j-1} \frac{A^j}{j!}.$$

The last expression, however, is

$$\sum_{j=1}^{\infty} j t^{j-1} \frac{A^j}{j!} = A \sum_{j=1}^{\infty} t^{j-1} \frac{A^{j-1}}{(j-1)!} = A e^{tA}.$$

A similar calculation, pulling out A to the back, yields that the sum is also $e^{tA}A$.

(c) Assume first that $AB = BA$. Repeated applications of this rule show $A^j B^k = B^k A^j$ for all $j, k \geqslant 0$. Thus,

$$e^{tA} e^{tB} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{t^j A^j}{j!} \frac{t^k B^k}{k!} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{t^j t^k A^j B^k}{j! k!} = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{t^k t^j B^k A^j}{k! j!} = e^{tB} e^{tA}.$$

Conversely, if $e^{tA} e^{tB} = e^{tB} e^{tA}$ for all $t > 0$, we get

$$\lim_{t\to 0} \frac{e^{tA} - \mathrm{id}}{t} \frac{e^{tB} - \mathrm{id}}{t} = \lim_{t\to 0} \frac{e^{tB} - \mathrm{id}}{t} \frac{e^{tA} - \mathrm{id}}{t}$$

and this proves $AB = BA$.

<u>Alternative solution for the converse:</u> If $s = j/n$ and $t = k/n$ for some common denominator n, we get from $e^{tA} e^{tB} = e^{tB} e^{tA}$ that

$$e^{tA} e^{sB} = \underbrace{e^{\frac{1}{n}A} \cdots e^{\frac{1}{n}A}}_{k} \underbrace{e^{\frac{1}{n}B} \cdots e^{\frac{1}{n}B}}_{j} = \underbrace{e^{\frac{1}{n}B} \cdots e^{\frac{1}{n}B}}_{j} \underbrace{e^{\frac{1}{n}A} \cdots e^{\frac{1}{n}A}}_{k} = e^{sB} e^{tA}.$$

Thus, if $s, t > 0$ are dyadic numbers, we get

$$A e^{sB} = \lim_{t\to 0} \frac{e^{tA} - \mathrm{id}}{t} e^{sB} = e^{sB} \lim_{t\to 0} \frac{e^{tA} - \mathrm{id}}{t} = e^{sB} A$$

and,

$$AB = A \lim_{s\to 0} \frac{e^{sB} - \mathrm{id}}{s} = \lim_{s\to 0} \frac{e^{sB} - \mathrm{id}}{s} A = BA.$$

(d) We have
$$e^{A/k} = \mathrm{id} + \tfrac{1}{k}\,A + \rho_k \quad\text{and}\quad k^2\rho_k = \sum_{j=2}^\infty \frac{A^j}{j!}\,\frac{1}{k^{j-2}}.$$
Note that $k^2\rho_k$ is bounded. Do the same for B (with the remainder term ρ_k') and multiply these expansions to get
$$e^{A/k}e^{B/k} = \mathrm{id} + \tfrac{1}{k}\,A + \tfrac{1}{k}\,B + \sigma_k$$
where $k^2\sigma_k$ is again bounded. In particular, if $k \gg 1$,
$$\left\|\tfrac{1}{k}\,A + \tfrac{1}{k}\,B + \sigma_k\right\| < 1.$$
This allows us to (formally) apply the logarithm series
$$\log(e^{A/k}e^{B/k}) = \tfrac{1}{k}\,A + \tfrac{1}{k}\,B + \sigma_k + \sigma_k'$$
where $k^2\sigma_k'$ is bounded. Multiply with k to get
$$k\,\log(e^{A/k}e^{B/k}) = A + B + \tau_k$$
with $k\tau_k$ bounded. Then we get
$$\begin{aligned}
e^{A+B} &= \lim_{k\to\infty} e^{A+B+\tau_k}\\
&= \lim_{k\to\infty} e^{k\log(e^{A/k}e^{B/k})}\\
&= \lim_{k\to\infty} \left(e^{\log(e^{A/k}e^{B/k})}\right)^k\\
&= \lim_{k\to\infty} \left(e^{A/k}e^{B/k}\right)^k
\end{aligned}$$

<u>Alternative Solution:</u> Set $S_k = e^{(A+B)/k}$ and $T_k = e^{A/k}e^{B/k}$. Then
$$S_k^k - T_k^k = \sum_{j=0}^{k-1} S_k^j(S_k - T_k)T_k^{k-1-j}.$$
This shows that
$$\begin{aligned}
\left\|S_k^k - T_k^k\right\| &\leqslant \sum_{j=0}^{k-1}\left\|S_k^j(S_k - T_k)T_k^{k-1-j}\right\|\\
&\leqslant \sum_{j=0}^{k-1}\left\|S_k^j\right\|\cdot\|S_k - T_k\|\cdot\left\|T_k^{k-1-j}\right\|\\
&\leqslant k\,\|S_k - T_k\|\cdot\max\{\|S_k\|,\|T_k\|\}^{k-1}\\
&\leqslant k\,\|S_k - T_k\|\cdot e^{\|A\|+\|B\|}.
\end{aligned}$$
Observe that
$$\|S_k - T_k\| = \left\|\sum_{j=0}^\infty \frac{(A+B)^j}{k^j\,j!} - \sum_{j=0}^\infty\sum_{l=0}^\infty \frac{A^j}{k^j\,j!}\,\frac{B^l}{k^l\,l!}\right\| \leqslant \frac{C}{k^2}$$
with a constant C depending only on $\|A\|$ and $\|B\|$. This yields $S_k^k - T_k^k \to 0$.

Problem 7.10 (Solution) (a) Let $0 < s < t$ and assume throughout that $h \in \mathbb{R}$ is such that $t - s - h > 0$. We have

$$
\begin{aligned}
&P_{t-(s+h)}T_{s+h} - P_{t-s}T_s \\
&\qquad = P_{t-(s+h)}T_{s+h} - P_{t-(s+h)}T_s + P_{t-(s+h)}T_s - P_{t-s}T_s \\
&\qquad = P_{t-(s+h)}(T_{s+h} - T_s) + (P_{t-(s+h)} - P_{t-s})T_s \\
&\qquad = (P_{t-(s+h)} - P_{t-s})(T_{s+h} - T_s) + P_{t-s}(T_{s+h} - T_s) + (P_{t-(s+h)} - P_{t-s})T_s.
\end{aligned}
$$

Divide by $h \neq 0$ to get for all $u \in \mathfrak{D}(A) \cap \mathfrak{D}(B)$

$$
\begin{aligned}
&\frac{1}{h}\Big(P_{t-(s+h)}T_{s+h}u - P_{t-s}T_s u\Big) \\
&\quad = (P_{t-(s+h)} - P_{t-s})\frac{T_{s+h}u - T_s u}{h} + P_{t-s}\frac{T_{s+h}u - T_s u}{h} + \frac{P_{t-(s+h)} - P_{t-s}}{h}T_s u \\
&\quad = \mathsf{I} + \mathsf{II} + \mathsf{III}.
\end{aligned}
$$

Letting $h \to 0$ gives

$$\mathsf{II} \to P_{t-s}BT_s \quad \text{and} \quad \mathsf{III} \to -P_{t-s}AT_s.$$

Let us show that $\mathsf{I} \to 0$. We have

$$\mathsf{I} = (P_{t-(s+h)} - P_{t-s})\left(\frac{T_{s+h}u - T_s u}{h} - T_s Bu\right) + (P_{t-(s+h)} - P_{t-s})T_s Bu = \mathsf{I}_1 + \mathsf{I}_2.$$

By the strong continuity of the semigroup $(P_t)_t$, we see that $\mathsf{I}_2 \to 0$ as $h \to 0$. Furthermore, by contractivity,

$$\|\mathsf{I}_1\| \leqslant \left(\left\|P_{t-(s+h)}\right\| + \|P_{t-s}\|\right) \cdot \left\|\frac{T_{s+h}u - T_s u}{h} - T_s Bu\right\| \leqslant 2\left\|\frac{T_{s+h}u - T_s u}{h} - T_s Bu\right\| \to 0$$

since $u \in \mathfrak{D}(B)$.

(b) In general, no. The problem is the semigroup property (unless T_t and P_s commute for all $s, t \geqslant 0$):

$$U_t U_s = T_t P_t T_s P_s \neq T_t T_s P_t P_s = T_{t+s}P_{t+s} = U_{t+s}.$$

In (c) we see how this can be ‘remedied’.

It is interesting to note (and helpful for the proof of (c)) that U_t is an operator on $\mathcal{C}_\infty$:

$$U_t : \mathcal{C}_\infty \xrightarrow{P_t} \mathcal{C}_\infty \xrightarrow{T_t} \mathcal{C}_\infty$$

and that U_t is strongly continuous: for all $s, t \geqslant 0$ and $f \in \mathcal{C}_\infty$

$$
\begin{aligned}
\|U_t f - U_s f\| &= \|T_t P_t f - T_s P_t f + T_s P_t f - T_s P_s f\| \\
&\leqslant \|(T_t - T_s)P_t f\| + \|T_s(P_t - P_s)f\| \\
&\leqslant \|(T_t - T_s)P_t f\| + \|(P_t - P_s)f\|
\end{aligned}
$$

and, as $s \to t$, both expressions tend to 0 since $f, P_t f \in \mathcal{C}_\infty$.

(c) Set $U_{t,n} := \left(T_{t/n}P_{t/n}\right)^n$.

<u>U_t is a contraction on $\mathcal{C}_\infty$:</u> By assumption, $P_{t/n}$ and $T_{t/n}$ map $\mathcal{C}_\infty$ into itself and, therefore, $T_{t/n}P_{t/n} : \mathcal{C}_\infty \to \mathcal{C}_\infty$ as well as $U_{t,n}$.

We have $\|U_{t,n}f\| = \left\|T_{t/n}P_{t/n}\cdots T_{t/n}P_{t/n}f\right\| \leqslant \prod_{j=1}^n \|T_{t/n}\|\|P_{t/n}\|\|f\| \leqslant \|f\|$. So, by the continuity of the norm

$$\|U_t f\| = \left\|\lim_n U_{t,n}f\right\| = \lim_n \|U_{t,n}f\| \leqslant \|f\|.$$

<u>Strong continuity:</u> Since the limit defining U_t is locally uniform in t, it is enough to show that $U_{t,n}$ is strongly continuous. Let X, Y be contractions in $\mathcal{C}_\infty$. Then we get

$$\begin{aligned} X^n - Y^n &= X^{n-1}X - X^{n-1}Y + X^{n-1}Y - Y^{n-1}Y \\ &= X^{n-1}(X - Y) + (X^{n-1} - Y^{n-1})Y \end{aligned}$$

hence, by the contraction property,

$$\|X^n f - Y^n f\| \leqslant \|(X - Y)f\| + \|(X^{n-1} - Y^{n-1})Yf\|.$$

By iteration, we get

$$\|X^n f - Y^n f\| \leqslant \sum_{k=0}^{n-1} \|(X - Y)Y^k f\|.$$

Take $Y = T_{t/n}P_{t/n}$, $X = T_{s/n}P_{s/n}$ where n is fixed. Then letting $s \to t$ shows the strong continuity of each $t \mapsto U_{t,n}$.

<u>Semigroup property:</u> Let $s, t \in \mathbb{Q}$ and write $s = j/m$ and $t = k/m$ for the same m. Then we take $n = l(j+k)$ and get

$$\begin{aligned} \left(T_{\frac{s+t}{n}}P_{\frac{s+t}{n}}\right)^n &= \left(T_{\frac{1}{lm}}P_{\frac{1}{lm}}\right)^{l(j+k)} \\ &= \left(T_{\frac{1}{lm}}P_{\frac{1}{lm}}\right)^{lj}\left(T_{\frac{1}{lm}}P_{\frac{1}{lm}}\right)^{lk} \\ &= \left(T_{\frac{j}{ljm}}P_{\frac{j}{ljm}}\right)^{lj}\left(T_{\frac{k}{lkm}}P_{\frac{k}{lkm}}\right)^{lk} \\ &= \left(T_{\frac{s}{lj}}P_{\frac{s}{lj}}\right)^{lj}\left(T_{\frac{t}{lk}}P_{\frac{t}{lk}}\right)^{lk} \end{aligned}$$

Since $n \to \infty \iff l \to \infty \iff lk, lj \to \infty$, we see that $U_{s+t} = U_s U_t$ for rational s, t.

For arbitrary s, t the semigroup property follows by approximation and the strong continuity of U_t: let $\mathbb{Q} \ni s_n \to s$ and $\mathbb{Q} \ni t_n \to t$. Then, by the contraction property,

$$\begin{aligned} \|U_s U_t f - U_{s_n}U_{t_n}f\| &\leqslant \|U_s U_t f - U_s U_{t_n}f\| + \|U_s U_{t_n}f - U_{s_n}U_{t_n}f\| \\ &\leqslant \|U_t f - U_{t_n}f\| + \|(U_s - U_{s_n})(U_{t_n} - U_t)f\| + \|(U_s - U_{s_n})U_t f\| \\ &\leqslant \|U_t f - U_{t_n}f\| + 2\|(U_{t_n} - U_t)f\| + \|(U_s - U_{s_n})U_t f\| \end{aligned}$$

and the last expression tends to 0. The limit $\lim_n U_{s_n+t_n}u = U_{s+t}u$ is obvious.

Generator: Let us begin with a heuristic argument (by ? and ?? indicate the steps which are questionable!). By the chain rule

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} U_t g &= \left.\frac{d}{dt}\right|_{t=0} \lim_n (T_{t/n}P_{t/n})^n g \\
&\overset{?}{=} \lim_n \left.\frac{d}{dt}\right|_{t=0} (T_{t/n}P_{t/n})^n g \\
&\overset{??}{=} \lim_n \left[n\big(T_{t/n}P_{t/n}\big)^{n-1}\big(T_{t/n}\tfrac{1}{n}BP_{t/n} + T_{t/n}\tfrac{1}{n}AP_{t/n}\big)g\Big|_{t=0}\right] \\
&= Bg + Ag.
\end{aligned}$$

So it is sensible to assume that $\mathfrak{D}(A)\cap\mathfrak{D}(B)$ is not empty. For the rigorous argument we have to justify the steps marked by question marks.

$\boxed{??}$ We have to show that $\frac{d}{ds}T_sP_sf$ exists and is $T_sAf + BP_sf$ for $f \in \mathfrak{D}(A)\cap\mathfrak{D}(B)$. This follows similar to (a) since we have for $s, h > 0$

$$\begin{aligned}
T_{s+h}P_{s+h}f - T_sP_sf &= T_{s+h}(P_{s+h} - P_s)f + (T_{s+h} - T_s)P_sf \\
&= (T_{s+h} - T_s)(P_{s+h} - P_s)f + T_s(P_{s+h} - P_s)f + (T_{s+h} - T_s)P_sf.
\end{aligned}$$

Divide by h. Then the first term converges to 0 as $h \to 0$, while the other two terms tend to T_sAf and BP_sf, respectively.

$\boxed{?}$ This is a matter of interchanging limit and differentiation. Recall the following theorem from calculus, e.g. Rudin [9, Theorem 7.17].

Theorem. *Let $(f_n)_n$ be a sequence of differentiable functions on $[0,\infty)$ which converges for some $t_0 > 0$. If $(f_n')_n$ converges [locally] uniformly, then $(f_n)_n$ converges [locally] uniformly to a differentiable function f and we have $f' = \lim_n f_n'$.*

This theorem holds for functions with values in any Banach space space and, therefore, we can apply it to the situation at hand: Fix $g \in \mathfrak{D}(A)\cap\mathfrak{D}(B)$; we know that $f_n(t) := U_{t,n}g$ converges (even locally uniformly) and, because of $\boxed{??}$, that $f_n'(t) = (T_{t/n}P_{t/n})^{n-1}(T_{t/n}A + BP_{t/n})g$.

Since $\lim_n(T_{t/n}P_{t/n})^n u$ converges locally uniformly, so does $\lim_n(T_{t/n}P_{t/n})^{n-1}u$; moreover, by the strong continuity, $T_{t/n}A + BP_{t/n}) \to (A+B)g$ locally uniformly for $g \in \mathfrak{D}(A)\cap\mathfrak{D}(B)$. Therefore, the assumptions of the theorem are satisfied and we may interchange the limits in the calculation above.

Problem 7.11 (Solution) The idea is to show that $A = -\frac{1}{2}\Delta$ is closed when defined on $\mathcal{C}^2_\infty(\mathbb{R})$. Since $\mathcal{C}^2_\infty(\mathbb{R}) \subset \mathfrak{D}(A)$ and since $(A, \mathfrak{D}(A))$ is the smallest closed extension, we are done. So let $(u_n)_n \subset \mathcal{C}^2_\infty(\mathbb{R})$ be a sequence such that $u_n \to u$ uniformly and $(Au_n)_n$ is a $\mathcal{C}_\infty$ Cauchy sequence. Since $\mathcal{C}_\infty(\mathbb{R})$ is complete, we can assume that $u_n'' \to 2g$ uniformly for some $g \in \mathcal{C}_\infty(\mathbb{R}^d)$. The aim is to show that $u \in \mathcal{C}^2_\infty$.

(a) By the fundamental theorem of differential and integral calculus we get

$$u_n(x) - u_n(0) - xu_n'(0) = \int_0^x (u_n'(y) - u_n'(0))\,dy = \int_0^x \int_0^y u_n''(z)\,dz.$$

Since $u_n'' \to 2g$ uniformly, we get

$$u_n(x) - u_n(0) - xu_n'(0) = \int_0^x \int_0^y u_n''(z)\,dz \to \int_0^x \int_0^y 2g(z)\,dz.$$

Since $u_n(x) \to u(x)$ and $u_n(0) \to u(0)$, we conclude that $u_n'(0) \to c$ converges.

(b) Recall the following theorem from calculus, e.g. Rudin [9, Theorem 7.17].

Theorem. *Let $(f_n)_n$ be a sequence of differentiable functions on $[0,\infty)$ which converges for some $t_0 > 0$. If $(f_n')_n$ converges uniformly, then $(f_n)_n$ converges uniformly to a differentiable function f and we have $f' = \lim_n f_n'$.*

If we apply this with $f_n' = u_n'' \to 2g$ and $f_n(0) = u_n'(0) \to c$, we get that $u_n'(x) - u_n'(0) \to \int_0^x 2g(z)\,dt$.

Let us determine the constant $c' := \lim_n u_n'(0)$. Since u_n' converges uniformly, the limit as $n \to \infty$ is in $\mathcal{C}_\infty$, and so we get

$$-\lim_{n\to\infty} u_n'(0)) = \lim_{x\to-\infty} \lim_{n\to\infty} (u_n'(x) - u_n'(0)) = \lim_{x\to-\infty} \int_0^x 2g(z)\,dz$$

i.e. $c' = \int_{-\infty}^0 g(z)\,dz$. We conclude that $u_n'(x) \to \int_{-\infty}^x g(z)\,dt$ uniformly.

(c) Again by the Theorem quoted in (b) we get $u_n(x) - u_n(0) \to \int_0^x \int_{-\infty}^y 2g(z)\,dz$ uniformly, and with the same argument as in (b) we get $u_n(0) = \int_{-\infty}^0 \int_{-\infty}^y 2g(z)\,dz$.

Problem 7.12 (Solution) By definition, (for all $\alpha > 0$ and formally but justifiable via monotone convergence also for $\alpha = 0$)

$$\begin{aligned} U_\alpha \mathbb{1}_C(x) &= \int_0^\infty e^{-\alpha t} P_t \mathbb{1}_C(x)\,dt \\ &= \int_0^\infty e^{-\alpha t}\, \mathbb{E}\, \mathbb{1}_C(B_t + x)\,dt \\ &= \mathbb{E} \int_0^\infty e^{-\alpha t} \mathbb{1}_{C-x}(B_t)\,dt. \end{aligned}$$

This is the 'discounted' (with 'interest rate' α) total amount of time a Brownian motion spends in the set $C - x$.

Problem 7.13 (Solution) <u>First formula:</u> We use induction. The induction start with $n = 0$ is clearly correct. Let us assume that the formula holds for some n and we do the induction step $n \rightsquigarrow n+1$. We have for $\beta \neq \alpha$

$$\begin{aligned} \frac{d^{n+1}}{d\alpha^{n+1}} U_\alpha f(x) &= \lim_{\beta\to\alpha} \frac{\frac{d^n}{d\alpha^n} U_\alpha f(x) - \frac{d^n}{d\beta^n} U_\beta f(x)}{\beta - \alpha} \\ &= \lim_{\beta\to\alpha} \frac{n!(-1)^n U_\alpha^{n+1} f(x) - n!(-1)^n U_\beta^{n+1} f(x)}{\beta - \alpha} \end{aligned}$$

$$= n!(-1)^n \lim_{\beta\to\alpha} \frac{U_\alpha^{n+1} f(x) - U_\beta^{n+1} f(x)}{\beta - \alpha}$$

Using the identity $a^{n+1} - b^{n+1} = (a-b)\sum_{j=0}^n a^{n-j}b^j$ we get, since the resolvents commute,

$$\frac{U_\alpha^{n+1} f(x) - U_\beta^{n+1} f(x)}{\beta - \alpha} = \frac{U_\alpha - U_\beta}{\beta - \alpha} \sum_{j=0}^n U_\alpha^{n-j} U_\beta^j f(x) = -U_\alpha U_\beta \sum_{j=0}^n U_\alpha^{n-j} U_\beta^j f(x)$$

In the last line we used the resolvent identity. Now we can let $\beta \to \alpha$ to get

$$\xrightarrow{\beta\to\alpha} -U_\alpha U_\alpha \sum_{j=0}^n U_\alpha^{n-j} U_\alpha^j f(x) = -(n+1) U_\alpha^{n+2} f(x).$$

This finishes the induction step.

<u>Second formula:</u> We use Leibniz' formula for the derivative of a product:

$$\partial^n(fg) = \sum_{j=0}^n \binom{n}{j} \partial^j f \partial^{n-j} g$$

and we get, using the first formula

$$\begin{aligned}
\partial^n(\alpha U_\alpha f(x)) &= \binom{n}{0} \alpha \partial^n U_\alpha f(x) + \binom{n}{1} \partial^{n-1} U_\alpha f(x) \\
&= \alpha n!(-1)^n U_\alpha^{n+1} f(x) + n(n-1)!(-1)^{n-1} U_\alpha^n f(x) \\
&= n!(-1)^{n+1} (\mathrm{id} - \alpha U_\alpha) U_\alpha^n f(x).
\end{aligned}$$

Problem 7.14 (Solution) (a) Let $f \geqslant 0$ be a Borel function. Then we get by monotone convergence

$$Uf(x) = \lim_{\alpha\to 0} U_\alpha f(x) = \lim_{\alpha \to 0} \int_0^\infty e^{-\alpha t} P_t f(x)\,dt = \int_0^\infty P_t f(x)\,dt.$$

Since $U_\alpha f = (\alpha\,\mathrm{id} - A)^{-1} f$, this calculation also shows that

$$Nf(x) = Uf(x) = \int_0^\infty P_t f(x)\,dt$$

for all positive, measurable $f \geqslant 0$. By the linearity of N, U and P_t, this equality follows for all measurable f if $Nf^\pm$, $Uf^\pm$ are finite.

(b) Let $g_t(x) = (2\pi t)^{-d/2} \exp(-|x|^2/(2t))$. Then by part a) we get

$$\begin{aligned}
g(x) &= \int_0^\infty g_t(x)\,dt \\
&= \int_0^\infty (2\pi t)^{-d/2} \exp(-|x|^2/(2t))\,dt \\
&\underset{dt = -|x|^2/(2s^2)\,ds}{\overset{s=|x|^2/(2t)}{=}} \int_0^\infty (2\pi)^{-d/2} \left(\frac{2s}{|x|^2}\right)^{d/2} e^{-s} \frac{|x|^2}{2s^2}\,ds \\
&= |x|^{2-d} (2\pi)^{-d/2} 2^{d/2-1} \int_0^\infty s^{d/2-2} e^{-s}\,ds \\
&= |x|^{2-d} \pi^{-d/2} \tfrac{1}{2} \Gamma\big(\tfrac{d}{2} - 1\big) \\
&= \frac{|x|^{2-d} \Gamma\big(\frac{d-2}{2}\big)}{2\,\pi^{d/2}}
\end{aligned}$$

$$= \frac{|x|^{2-d}\,\frac{d-2}{2}\,\Gamma\left(\frac{d-2}{2}\right)}{2\,\pi^{d/2}\,\frac{d-2}{2}}$$
$$= \frac{|x|^{2-d}\,\Gamma\left(\frac{d}{2}\right)}{\pi^{d/2}\,(d-2)}.$$

Problem 7.15 (Solution) (a) The process (t, B_t) starts at $(0, B_0) = 0$, and if we start at (s, x) we consider the process $(s + t, x + B_t) = (s, x) + (t, B_t)$. Let $f \in \mathcal{B}_b([0, \infty) \times \mathbb{R})$. Since the motion in t is deterministic, we can use the probability space $(\Omega, \mathcal{A}, \mathbb{P} = \mathbb{P})$ generated by the Brownian motion $(B_t)_{t\geqslant 0}$. Then

$$T_t f(s,x) := \mathbb{E}^{(s,x)} f(t, B_t) := \mathbb{E} f(s+t, x+B_t).$$

<u>T_t preserves $\mathcal{C}_\infty([0,\infty)\times\mathbb{R})$:</u> If $f \in \mathcal{C}_\infty([0,\infty)\times\mathbb{R})$, we see with dominated convergence that

$$\begin{aligned}\lim_{(\sigma,\xi)\to(s,x)} T_t f(\sigma,\xi) &= \lim_{(\sigma,\xi)\to(s,x)} \mathbb{E} f(\sigma+t, \xi+B_t)\\ &= \mathbb{E} \lim_{(\sigma,\xi)\to(s,x)} f(\sigma+t, \xi+B_t)\\ &= \mathbb{E} f(s+t, x+B_t)\\ &= T_t f(s,x)\end{aligned}$$

which shows that T_t preserves $f \in \mathcal{C}_b([0,\infty)\times\mathbb{R})$. In a similar way we see that

$$\lim_{|(\sigma,\xi)|\to\infty} T_t f(\sigma,\xi) = \mathbb{E} \lim_{|(\sigma,\xi)|\to\infty} f(\sigma+t, \xi+B_t) = 0,$$

i.e. T_t maps $\mathcal{C}_\infty([0,\infty)\times\mathbb{R})$ into itself.

<u>T_t is a semigroup:</u> Let $f \in \mathcal{C}_\infty([0,\infty)\times\mathbb{R})$. Then, by the independence and stationary increments property of Brownian motion,

$$\begin{aligned}T_{t+\tau} f(s,x) &= \mathbb{E} f(s+t+\tau, x+B_{t+\tau})\\ &= \mathbb{E} f(s+t+\tau, x+(B_{t+\tau}-B_t)+B_t)\\ &= \mathbb{E}\,\mathbb{E}^{(t,B_t)} f(s+\tau, x+(B_{t+\tau}-B_t))\\ &= \mathbb{E}\,\mathbb{E}^{(t,B_t)} f(s+\tau, x+(B_\tau)\\ &= \mathbb{E}\, T_\tau(s+t, x+B_t)\\ &= T_t T_\tau(s,x).\end{aligned}$$

<u>T_t is strongly continuous:</u> Since $f \in \mathcal{C}_\infty([0,\infty)\times\mathbb{R})$ is uniformly continuous, we see that for every $\epsilon > 0$ there is some $\delta > 0$ such that

$$|f(s+h, x+y) - f(s,x)| \leqslant \epsilon \quad \forall h + |y| \leqslant 2\delta.$$

So, let $t < h < \delta$, then

$$|T_t f(s,x) - f(s,x)| = \left|\mathbb{E}\left(f(s+t, x+B_t) - f(s,x)\right)\right|$$

$$\leqslant \int_{|B_t|\leqslant\delta} |f(s+t,x+B_t)-f(s,x)|\,d\mathbb{P} + 2\|f\|_\infty\,\mathbb{P}(|B_t|>\delta)$$
$$\leqslant \epsilon + 2\|f\|_\infty\,\frac{1}{\delta^2}\,\mathbb{E}(B_t^2)$$
$$= \epsilon + 2\|f\|_\infty\,\frac{t}{\delta^2}.$$

Since the estimate is uniform in (s,x), this proves strong continuity.

<u>Markov property:</u> this is trivial.

(b) The transition semigroup is

$$T_t f(s,x) = \mathbb{E}\, f(s+t,x+B_t) = (2\pi t)^{-1/2}\int_{\mathbb{R}} f(s+t,x+y)\,e^{-y^2/(2t)}\,dy.$$

The resolvent is given by

$$U_\alpha f(s,x) = \int_0^\infty e^{-t\alpha} T_t f(s,x)\,dt$$

and the generator is, for all $f\in\mathcal{C}^{1,2}([0,\infty)\times\mathbb{R})$

$$\begin{aligned}\frac{T_t f(s,x)-f(s,x)}{t} &= \frac{\mathbb{E}\, f(s+t,x+B_t)-f(s,x)}{t}\\ &= \frac{\mathbb{E}\, f(s+t,x+B_t)-f(s,x+B_t)}{t} + \frac{\mathbb{E}\, f(s,x+B_t)-f(s,x)}{t}\\ &\xrightarrow{t\to 0} \mathbb{E}\,\partial_t f(s,x+B_0) + \tfrac12\Delta_x f(s,x)\\ &= (\partial_t + \tfrac12\,\Delta_x) f(s,x).\end{aligned}$$

Note that, in view of Theorem 7.19, pointwise convergence is enough (provided the pointwise limit is a $\mathcal{C}_\infty$-function).

(c) We get for $u\in\mathcal{C}^{1,2}_\infty$ that under $\mathbb{P}^{(s,x)}$

$$M_t^u := u(s+t,x+B_t) - u(s,x) - \int_0^t \left(\partial_r + \tfrac12\,\Delta_x\right) u(s+r,x+B_r)\,dr$$

is an $\mathcal{F}_t$-martingale. This is the same assertion as in Theorem 5.6 (up to the choice of u which is restricted here as we need it in the domain of the generator...).

Problem 7.16 (Solution) Let $u\in\mathfrak{D}(A)$ and σ a stopping time with $\mathbb{E}\,\sigma<\infty$. Use optional stopping (Theorem A.18 in combination with remark A.21) to see that

$$M_{\sigma\wedge t}^u := u(X_{\sigma\wedge t}) - u(x) - \int_0^{\sigma\wedge t} Au(X_r)\,dr$$

is a martingale (for either $\mathcal{F}_t$ or $\mathcal{F}_{\sigma\wedge t}$). If we take expectations we get

$$\mathbb{E}^x\, u(X_{\sigma\wedge t}) - u(x) = \mathbb{E}^x\left(\int_0^{\sigma\wedge t} Au(X_r)\,dr\right).$$

Since $u, Au\in\mathcal{C}_\infty$ we see

$$\left|\mathbb{E}^x\left(\int_0^{\sigma\wedge t} Au(X_r)\,dr\right)\right| \leqslant \mathbb{E}^x\left(\int_0^{\sigma\wedge t}\|Au\|_\infty\,dr\right) \leqslant \|Au\|_\infty\cdot\mathbb{E}^x\,\sigma<\infty,$$

i.e. we can use dominated convergence and let $t\to\infty$. Because of the right-continuity of the paths of a Feller process we get Dynkin's formula (7.21).

Problem 7.17 (Solution) Clearly,

$$\mathbb{P}(X_t \in F \quad \forall t \in \mathbb{R}^+) \leqslant \mathbb{P}(X_q \in F \quad \forall q \in \mathbb{Q}^+).$$

On the other hand, since F is closed and X_t has continuous paths,

$$X_q \in F \quad \forall q \in \mathbb{Q}^+ \implies X_t = \lim_{\mathbb{Q}^+ \ni q \to t} X_q \in F \quad \forall t \geqslant 0$$

and the converse inequality follows.

8 The PDE Connection

Problem 8.1 (Solution) Write $g_t(x) = (2\pi t)^{-d/2} e^{-|x|^2/2t}$ for the heat kernel. Since convolutions are smoothing, one finds easily that $P_\epsilon f = g_\epsilon \star f \in \mathcal{C}_\infty^\infty \subset \mathfrak{D}(\Delta)$. (There is a more general concept behind it: whenever the semigroup is *analytic*—i.e. $z \mapsto P_z$ has an extension to, say, a sector in the complex plane and it is holomorphic there—one has that T_t maps the underlying Banach space into the domain of the generator; cf. e.g. Pazy [6, pp. 60–63].)

Thus, if we set $f_\epsilon := P_\epsilon f$, we can apply Lemma 8.1 and find that

$$u_\epsilon(t,x) \overset{\text{Lemma 8.1}}{=} P_t f_\epsilon(x) \overset{\text{def}}{=} P_t P_\epsilon f(x) \underset{\text{group}}{\overset{\text{semi-}}{=}} P_{t+\epsilon} f(x).$$

By the strong continuity of the heat semigroup, we find that

$$u_\epsilon(t,x) \xrightarrow[\epsilon\to 0]{\text{uniformly}} P_t f(x).$$

Moreover,

$$\begin{aligned}\frac{\partial}{\partial t} u_\epsilon(t,\cdot) &= \frac{1}{2}\Delta_x P_t P_\epsilon f \\ &= P_\epsilon \underbrace{\Big(\frac{1}{2}\Delta_x P_t f\Big)}_{\in \mathcal{C}_\infty} \xrightarrow[\epsilon\to 0]{\text{uniformly}} \frac{1}{2}\Delta_x P_t f.\end{aligned}$$

Since both the sequence and the differentiated sequence converge uniformly, we can interchange differentiation and the limit, cf. [9, Theorem 7.17, p. 152], and we get

$$\frac{\partial}{\partial t} u(t,x) = \lim_{\epsilon\to 0}\frac{\partial}{\partial t} u_\epsilon(t,x) = \frac{1}{2}\Delta_x u(t,x)$$

and

$$u_\epsilon(0,\cdot) = P_\epsilon f \xrightarrow[\epsilon\to 0]{} f = u(0,\cdot)$$

and we get a solution for the initial value f. The proof of the uniqueness in Lemma 8.1 stays valid.

Problem 8.2 (Solution) By differentiation we get $\frac{d}{dt}\int_0^t f(B_s)\,ds = f(B_t)$ so that $f(B_t) = 0$. We can assume that f is positive and bounded, otherwise we could consider $f^\pm(B_t) \wedge c$ for some constant $c > 0$. Now $\mathbb{E} f(B_t) = 0$ and we conclude from this that $f = 0$.

Problem 8.3 (Solution) a) Note that

$$\Big|\chi_n(B_t) e^{-\alpha\int_0^t g_n(B_s)\,ds}\Big| \leqslant \Big|e^{-\alpha\int_0^t ds}\Big| = e^{-\alpha t} \leqslant 1$$

is uniformly bounded. Moreover,

$$\lim_{n\to\infty} \chi_n(B_t) e^{-\alpha\int_0^t g_n(B_s)\,ds} = \mathbb{1}_{\mathbb{R}}(B_t) e^{-\alpha\int_0^t \mathbb{1}_{(0,\infty)}(B_s)\,ds}$$

which means that, by dominated convergence,

$$v_{n,\lambda}(x) = \int_0^\infty e^{-\lambda t}\, \mathbb{E}\left(\chi_n(B_t) e^{-\alpha\int_0^t g_n(B_s)\,ds}\right) dt \xrightarrow[n\to\infty]{} v_\lambda(x).$$

Moreover, we get that $|v_\lambda(x)| \leqslant \lambda^{-1}$.

If we rearrange (8.12) we see that

$$v''_{n,\lambda}(x) = 2(\alpha\chi_n(x) + \lambda) v_{n,\lambda}(x) - g_n(x), \tag{*}$$

and since the expression on the right has a limit as $n \to \infty$, we get that $\lim_{n\to\infty} v''_{n,\lambda}(x)$ exists.

b) Integrating (*) we find

$$v'_{n,\lambda}(x) - v'_{n,\lambda}(0) = 2\int_0^x (\alpha\chi_n(y) + \lambda) v_{n,\lambda}(y)\,dy - \int_0^x g_n(y)\,dy, \tag{**}$$

and, again by dominated convergence, we conclude that $\lim_{n\to\infty}\left[v'_{n,\lambda}(x) - v'_{n,\lambda}(0)\right]$ exists. In addition, the right-hand side is uniformly bounded (for all $|x| \leqslant R$):

$$\begin{aligned}\left|2\int_0^x (\alpha\chi_n(y) + \lambda) v_{n,\lambda}(y)\,dy - \int_0^x g_n(y)\,dy\right| &\leqslant 2\int_0^R (\alpha + \lambda)\,dy + \int_0^R dy \\ &\leqslant 2(\alpha + \lambda + 1)R.\end{aligned}$$

Integrating (**) reveals

$$v_{n,\lambda}(x) - v_{n,\lambda}(0) - x\, v'_{n,\lambda}(0) = \int_0^x \left[v'_{n,\lambda}(z) - v'_{n,\lambda}(0)\right] dz.$$

Since the expression under the integral converges boundedly and since $\lim_{n\to\infty} v_{n,\lambda}(x)$ exists, we conclude that $\lim_{n\to\infty} v'_{n,\lambda}(0)$ exists. Consequently, $\lim_{n\to\infty} v'_{n,\lambda}(x)$ exists.

c) The above considerations show that

$$\begin{aligned} v_\lambda(x) &= \lim_{n\to\infty} v_{n,\lambda}(x) \\ v'_\lambda(x) &= \lim_{n\to\infty} v'_{n,\lambda}(x) \\ v''_\lambda(x) &= \lim_{n\to\infty} v''_{n,\lambda}(x) \end{aligned}$$

Problem 8.4 (Solution) We have to show that $v(t,x) := \int_0^t P_s g(x)\,ds$ is the unique solution of the initial value problem (8.7) with $g = g(x)$ satisfying $|v(t,x)| \leqslant C\,t$.

<u>Existence:</u> The linear growth bound is obvious from $|P_s g(x)| \leqslant \|P_s g\|_\infty \leqslant \|g\|_\infty < \infty$. The rest follows from the hint if we take $A = \frac{1}{2}\Delta$ and Lemma 7.10.

Uniqueness: We proceed as in the proof of Lemma 8.1. Set $v_\lambda(x) := \int_0^\infty e^{-\lambda t} v(t,x)\,dt$. This integral is, for $\lambda > 0$, convergent and it is the Laplace transform of $v(\cdot, x)$. Under the Laplace transform the initial value problem (8.7) with $g = g(x)$ becomes

$$\lambda v_\lambda(x) - Av_\lambda(x) = \lambda^{-1} g(x)$$

and this problem has a unique solution, cf. Proposition 7.13 f). Since the Laplace transform is invertible, we see that v is unique.

Problem 8.5 (Solution) Integrating $u''(x) = 0$ twice yields

$$u'(x) = c \quad \text{and} \quad u(x) = cx + d$$

with two integration constants $c, d \in \mathbb{R}$. The boundary conditions $u(0) = a$ and $u(1) = b$ show that

$$d = a \quad \text{and} \quad c = b - a$$

so that

$$u(x) = (b-a)x + a.$$

On the other hand, by Corollary 5.11 (Wald's identities), Brownian motion started in $x \in (0,1)$ has the probability to exit (at the exit time τ) the interval $(0,1)$ in the following way:

$$\mathbb{P}^x(B_\tau = 1) = x \quad \text{and} \quad \mathbb{P}^x(B_\tau = 0) = 1 - x.$$

Therefore, if $f : \{0,1\} \to \mathbb{R}$ is a function on the boundary of the interval $(0,1)$ such that $f(0) = a$ and $f(1) = b$,then

$$\mathbb{E}^x f(B_\tau) = (1-x)f(0) + xf(1) = (b-a)x + a.$$

This means that $u(x) = \mathbb{E}^x f(B_\tau)$, a result which we will see later in Section 8.4 in much greater generality.

Problem 8.6 (Solution) The key is to show that *all* points in the open and bounded, hence relatively compact, set D are non-absorbing. Thus the closure of D has an neighbourhood, say $V \supset \bar{D}$ such that $\mathbb{E}\,\tau_{D^c} \leqslant \mathbb{E}\,\tau_{V^c}$. Let us show that $\mathbb{E}\,\tau_{V^c} < \infty$.

Since D is bounded, there is some $R > 0$ such that $\mathbb{B}(0,R) \supset \bar{D}$. Pick some test function $\chi = \chi_R$ such that $\chi|_{\mathbb{B}^c(0,R)} \equiv 0$ and $\chi \in C_c^\infty(\mathbb{R}^d)$. Pick further some function $u \in \mathcal{C}^2(\mathbb{R}^d)$ such that $\Delta u > 0$ in $\mathbb{B}(0,2R)$. Here are two possibilities to get such a function:

$$u(x) = |x|^2 = \sum_{j=1}^d x_j^2 \implies \tfrac{1}{2}\Delta u(x) = 1$$

or, if $f \in \mathcal{C}_b(\mathbb{R}^d)$, $f \geqslant 0$ and $f = f(x_1)$ we set

$$F(x) = F(x_1) := \int_0^{x_1} f(z_1)\,dz_1$$

and

$$U(x) = U(x_1) := \int_0^{x_1} F(y_1)\,dy_1 = \int_0^{x_1}\int_0^{y_1} f(z_1)\,dz_1.$$

Clearly, $\frac{1}{2}\Delta U(x) = \frac{1}{2}\partial_{x_1}^2 U(x_1) = f(x_1)$, and we can arrange things by picking the correct f.

Problem: neither u nor U will be in $\mathfrak{D}(\Delta)$ (unless you are so lucky as in the proof of Lemma 8.8 to pick instantly the right function).

Now observe that

$$\begin{aligned}\chi\cdot u,\ \chi\cdot U &\in \mathcal{C}_c^2(\mathbb{R}^d)\subset\mathfrak{D}(\Delta)\\ \Delta(\chi\cdot U) &= \chi\cdot\Delta U + U\cdot\Delta\chi + 2\langle\nabla\chi,\ \nabla U\rangle\end{aligned}$$

which means that

$$\Delta(\chi\cdot U)\big|_{\mathbb{B}(0,R)} = \Delta U\big|_{\mathbb{B}(0,R)}.$$

The rest of the proof follows now either as in Lemma 7.24 or Lemma 8.8 (both employ, anyway, the same argument based on Dynkin's formula).

Problem 8.7 (Solution) We are following the hint. Let $L = \sum_{j,k=1}^d a_{jk}(x)\,\partial_j\partial_k + \sum_{j=1}^d b_j(x)\,\partial_j$. Then

$$\begin{aligned}L(\chi f) &= \sum_{j,k} a_{jk}\partial_j\partial_k(\chi f) + \sum_j b_j\partial_j(\chi f)\\ &= \sum_{j,k} a_{jk}\big(\partial_j\partial_k\chi + \partial_j\partial_k f + \partial_k\chi\partial_j f + \partial_j\chi\partial_k f\big) + \sum_j b_j\big(f\partial_j\chi + \chi\partial_j f\big)\\ &= \chi Lf + fL\chi + \sum_{j,k}(a_{jk}+a_{kj})\partial_j\chi\partial_k f.\end{aligned}$$

If $|x|<R$ and $\chi|_{\mathbb{B}(0,R)} = 1$, then $L(u\chi)(x) = Lu(x)$. Set $u(x) = e^{-x_1^2/\gamma r^2}$. Then only the derivatives in x_1-direction give any contribution and we get

$$\partial_1 u(x) = -\frac{2x_1}{\gamma r^2}\,e^{-\frac{x_1^2}{\gamma r^2}} \quad\text{and}\quad \partial_1^2 u(x) = \frac{2}{\gamma r^2}\left(\frac{2x_1^2}{\gamma r^2}-1\right)e^{-\frac{x_1^2}{\gamma r^2}}$$

Thus we get for $L(-u) = -Lu$ and any $|x|<r$

$$\begin{aligned}-Lu(x) &= \frac{2a_{11}(x)}{\gamma r^2}\left(1-\frac{2x_1^2}{\gamma r^2}\right)e^{-\frac{x_1^2}{\gamma r^2}} + \frac{2b_1(x)x_1}{\gamma r^2}\,e^{-\frac{x_1^2}{\gamma r^2}}\\ &= \left[\frac{2a_{11}(x)}{\gamma r^2}\left(1-\frac{2x_1^2}{\gamma r^2}\right)+\frac{2b_1(x)x_1}{\gamma r^2}\right]e^{-\frac{x_1^2}{\gamma r^2}}\\ &\geqslant \left[\frac{2a_0}{\gamma r^2}\left(1-\frac{2}{\gamma}\right)-\frac{2b_0}{\gamma r}\right]e^{-\frac{r^2}{\gamma r^2}}\end{aligned}$$

This shows that the drift $b_1(x)$ can make the expression in the bracket negative!

Let us modify the Ansatz. Observe that for $f(x) = f(x_1)$ we have

$$Lf(x) = a_{11}(x)\partial_1^2 f(x) - b_1(x)\partial_1 f(x)$$

and if we know that $\partial_1^2 f, \partial_1 f \geqslant 0$ we get

$$Lf(x) \geqslant a_0 \partial_1^2 f(x) - b_0 \partial_1 f(x) \overset{!!}{>} 0.$$

This means that $\partial_1^2 f / \partial_1 f > b_0/a_0$ seems to be natural and a reasonable Ansatz would be

$$f(x) = \int_0^{x_1} e^{\frac{2b_0}{a_0} y}\, dy.$$

Then

$$\partial_1 f(x) = e^{\frac{2b_0}{a_0} x_1} \quad \text{and} \quad \partial_1^2 f(x) = \frac{2b_0}{a_0} e^{\frac{2b_0}{a_0} x_1}$$

and we get

$$\begin{aligned} Lf(x) &= a_{11}(x) \frac{2b_0}{a_0} e^{\frac{2b_0}{a_0} x_1} - b_1(x) e^{\frac{2b_0}{a_0} x_1} \\ &\geqslant a_0 \frac{2b_0}{a_0} e^{\frac{2b_0}{a_0} x_1} - b_0\, e^{\frac{2b_0}{a_0} x_1} \\ &\geqslant (2b_0 - b_0)\, e^{\frac{2b_0}{a_0} x_1} > 0. \end{aligned}$$

With the above localization trick on balls, we are done.

Problem 8.8 (Solution) Assume that $B_0 = 0$. Any other starting point can be reduced to this situation by shifting Brownian motion to $B_0 = 0$. The LIL shows that a Brownian motion satisfies

$$-1 = \varliminf_{t \to 0} \frac{B(t)}{\sqrt{2t \log\log \frac{1}{t}}} < \varlimsup_{t \to 0} \frac{B(t)}{\sqrt{2t \log\log \frac{1}{t}}} = 1$$

i.e. $B(t)$ oscillates for $t \to 0$ between the curves $\pm\sqrt{2t \log\log \frac{1}{t}}$. Since a Brownian motion has continuous sample paths, this means that it has to cross the level 0 infinitely often.

Problem 8.9 (Solution) The idea is to proceed as in Example 8.12 e) where Zaremba's needle plays the role a truncated flat cone in dimension $d = 2$ (but in dimension $d \geqslant 3$ it has too small dimension). The set-up is as follows: without loss of generality we take $x_0 = 0$ (otherwise we shift Brownian motion) and we assume that the cone lies in the hyperplane $\{x \in \mathbb{R}^d : x_1 = 0\}$ (otherwise we rotate things).

Let $B(t) = (b(t), \beta(t))$, $t \geqslant 0$, be a BM^d where $b(t)$ is a BM^1 and $\beta(t)$ is a $(d-1)$-dimensional Brownian motion. Since B is a BM^d, we know that the coordinate processes $b = (b(t))_{t \geqslant 0}$ and $\beta = (\beta(t))_{t \geqslant 0}$ are independent processes. Set $\sigma_n = \inf\{t > 1/n : b(t) = 0\}$. Since $0 \in \mathbb{R}$ is regular for $\{0\} \subset \mathbb{R}$, see Example 8.12 e), we get that $\lim_{n \to \infty} \sigma_n = \tau_{\{0\}} = 0$ almost surely with respect to $\mathbb{P}^0$. Since $\beta \perp\!\!\!\perp b$, the random variable $\beta(\sigma_n)$ is rotationally symmetric (see, e.g., the solution to Problem 8.10).

Let C be a flat (i.e. in the hyperplane $\{x \in \mathbb{R}^d : x_1 = 0\}$) cone such that some truncation C' of it lies in D^c. By rotational symmetry, we get

$$\mathbb{P}^0(\beta(\sigma_n) \in C) = \gamma = \frac{\text{opening angle of } C}{\text{full angle}}.$$

By continuity of BM, $\beta(\sigma_n) \to \beta(0) = 0$, and this gives

$$\mathbb{P}^0(\beta(\sigma_n) \in C') = \gamma.$$

Clearly, $B(\sigma_n) = (b(\sigma_n), \beta(\sigma_n)) = (0, \beta(\sigma_n))$ and $\{\beta(\sigma_n) \in C'\} \subset \{\tau_{D^c} \leqslant \sigma_n\}$, so

$$\mathbb{P}^0(\tau_{D^c} = 0) = \lim_{n\to\infty} \mathbb{P}^0(\tau_{D^c} \leqslant \sigma_n) \geqslant \varlimsup_{n\to\infty} \mathbb{P}^0(\beta(\sigma_n) \in C') \geqslant \gamma > 0.$$

Now Blumenthal's 0-1–law, Corollary 6.22, applies and gives $\mathbb{P}^0(\tau_{D^c} = 0) = 1$.

Problem 8.10 (Solution) Proving that the random variable $\beta(\sigma_n)$ is absolutely continuous with respect to Lebesgue measure is relatively easy: note that, because of the independence of b and β, hence σ_n and β,

$$\begin{aligned}
-\frac{d}{dx}\,\mathbb{P}^0(\beta(\sigma_n) \geqslant x) &= -\frac{d}{dx}\int_{\mathbb{R}} \mathbb{P}^0(\beta_t \geqslant x)\;\mathbb{P}(\sigma_n \in dt)\\
&= \int_{\mathbb{R}} -\frac{d}{dx}\,\mathbb{P}^0(\beta_t \geqslant x)\;\mathbb{P}(\sigma_n \in dt)\\
&= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}}\,e^{-x^2/(2t)}\;\mathbb{P}(\sigma_n \in dt)\\
&= \int_{1/n}^{\infty} \frac{1}{\sqrt{2\pi t}}\,e^{-x^2/(2t)}\;\mathbb{P}(\sigma_n \in dt).
\end{aligned}$$

(observe, for the last equality, that σ_n takes values in $[1/n, \infty)$.) Since the integrand is bounded (even as $t \to 0$), the interchange of integration and differentiation is clearly satisfied.

<u>$(d-1)$-dimensional version:</u> Let β be a $(d-1)$-dimensional version as in Problem 8.9 Proving that the random variable $\beta(\sigma_n)$ is rotationally symmetric is easy: note that, because of the independence of b and β, hence σ_n and β, we have for all Borel sets $A \subset \mathbb{R}^{d-1}$

$$\mathbb{P}^0(\beta(\sigma_n) \in A) = \int_{1/n}^{\infty} \mathbb{P}^0(\beta_t \in A)\;\mathbb{P}(\sigma_n \in dt)$$

and this shows that the rotational symmetry of β is inherited by $\beta(\sigma_n)$.

We even get a density by formally replacing A by dx:

$$\begin{aligned}
\beta(\sigma_n) &\sim \int_{\mathbb{R}} \mathbb{P}^0(\beta_t \in dx)\;\mathbb{P}(\sigma_n \in dt)\\
&= \int_{1/n}^{\infty} \frac{1}{(2\pi t)^{(d-1)/2}}\,e^{-|x|^2/(2t)}\;\mathbb{P}(\sigma_n \in dt)\,dx.
\end{aligned}$$

(here $x \in \mathbb{R}^{d-1}$).

It is a bit more difficult to work out the exact shape of the density. Let us first determine the distribution of σ_n. Clearly,

$$\{\sigma_n > t\} = \Big\{ \inf_{1/n \leqslant s \leqslant t} |b(s)| > 0 \Big\}.$$

By the Markov property of Brownian motion we get

$$\begin{aligned}
\mathbb{P}^0(\sigma_n > t) &= \mathbb{P}^0\Big(\inf_{1/n \leqslant s \leqslant t} |b(s)| > 0 \Big) \\
&= \mathbb{E}^0\, \mathbb{P}^{b(1/n)}\Big(\inf_{s \leqslant t-1/n} |b(s)| > 0 \Big) \\
&= \mathbb{E}^0\Big(\mathbb{1}_{\{b(1/n)>0\}}\, \mathbb{P}^{b(1/n)}\Big(\inf_{s \leqslant t-1/n} b(s) > 0 \Big) \\
&\qquad + \mathbb{1}_{\{b(1/n)<0\}}\, \mathbb{P}^{b(1/n)}\Big(\sup_{s \leqslant t-1/n} b(s) < 0 \Big)\Big) \\
&= \mathbb{E}^0\Big(\mathbb{1}_{\{b(1/n)>0\}}\, \mathbb{P}^{0}\Big(\inf_{s \leqslant t-1/n} b(s) > -y \Big) \\
&\qquad + \mathbb{1}_{\{b(1/n)<0\}}\, \mathbb{P}^{0}\Big(\sup_{s \leqslant t-1/n} b(s) < -y \Big)\Big|_{y=b(1/n)}\Big) \\
&\overset{b \sim -b}{=} \mathbb{E}^0\Big(\mathbb{1}_{\{b(1/n)>0\}}\, \mathbb{P}^{0}\Big(\sup_{s \leqslant t-1/n} b(s) < y \Big) \\
&\qquad + \mathbb{1}_{\{b(1/n)<0\}}\, \mathbb{P}^{0}\Big(\sup_{s \leqslant t-1/n} b(s) < -y \Big)\Big|_{y=b(1/n)}\Big) \\
&\overset{b \sim -b}{=} \mathbb{E}^0\Big(\mathbb{1}_{\{b(1/n)>0\}}\, \mathbb{P}^{0}\Big(\sup_{s \leqslant t-1/n} b(s) < y \Big) \\
&\qquad + \mathbb{1}_{\{b(1/n)>0\}}\, \mathbb{P}^{0}\Big(\sup_{s \leqslant t-1/n} b(s) < -y \Big)\Big|_{-y=b(1/n)}\Big) \\
&= 2\, \mathbb{E}^0\Big(\mathbb{1}_{\{b(1/n)>0\}}\, \mathbb{P}^{0}\Big(\sup_{s \leqslant t-1/n} b(s) < y \Big)\Big|_{y=b(1/n)}\Big) \\
&\overset{(6.12)}{=} 2\, \mathbb{E}^0\Big(\mathbb{1}_{\{b(1/n)>0\}}\, \mathbb{P}^{0}\Big(|b(t-1/n)| < y \Big)\Big|_{y=b(1/n)}\Big) \\
&= 4 \int_0^\infty \mathbb{P}^0\big(b(t-1/n) < y\big)\, \mathbb{P}^0(b(1/n) \in dy) \\
&= \frac{2}{\pi} \frac{1}{\sqrt{t-\frac{1}{n}}\sqrt{\frac{1}{n}}} \int_0^\infty \int_0^y e^{-z^2/2(t-1/n)}\, dz\, e^{-ny^2/2}\, dy
\end{aligned}$$

change of variables: $\zeta = z/\sqrt{t-\frac{1}{n}}$

$$= \frac{2\sqrt{n}}{\pi} \int_0^\infty \int_0^{y/\sqrt{t-\frac{1}{n}}} e^{-\zeta^2/2}\, d\zeta\, e^{-ny^2/2}\, dy.$$

For the density we differentiate in t:

$$\begin{aligned}
-\frac{d}{dt}\, \mathbb{P}^0(\sigma_n > t) &= -\frac{2\sqrt{n}}{\pi} \frac{d}{dt} \int_0^\infty \int_0^{y/\sqrt{t-\frac{1}{n}}} e^{-\zeta^2/2}\, d\zeta\, e^{ny^2/2}\, dy \\
&= \frac{\sqrt{n}}{\pi}\big(t-\tfrac{1}{n}\big)^{-3/2} \int_0^\infty y\, e^{-y^2/2(t-\frac{1}{n})}\, e^{-ny^2/2}\, dy \\
&= \frac{\sqrt{n}}{\pi}\big(t-\tfrac{1}{n}\big)^{-3/2} \int_0^\infty y\, e^{-\frac{y^2}{2}\frac{nt}{t-1/n}}\, dy \\
&= \frac{\sqrt{n}}{\pi}\big(t-\tfrac{1}{n}\big)^{-3/2} \frac{t-\frac{1}{n}}{nt} \left[-e^{-\frac{y^2}{2}\frac{nt}{t-1/n}} \right]_{y=0}^\infty \\
&= \frac{\sqrt{n}}{\pi}\big(t-\tfrac{1}{n}\big)^{-1/2} \frac{1}{nt}
\end{aligned}$$

$$= \frac{1}{\pi} \frac{1}{t\sqrt{nt-1}}.$$

Now we proceed with the d-dimensional case. We have for all $x \in \mathbb{R}^{d-1}$

$$\begin{aligned}
\beta(\sigma_n) &\sim \int_{1/n}^{\infty} \frac{1}{(2\pi t)^{(d-1)/2}}\, e^{-|x|^2/(2t)}\, \mathbb{P}(\sigma_n \in dt)\, dx \\
&= \frac{1}{\pi^{(d+1)/2} 2^{(d-1)/2}} \int_{1/n}^{\infty} \frac{1}{t^{(d+1)/2}\sqrt{nt-1}}\, e^{-|x|^2/(2t)}\, dt \\
&= \frac{n^{(d-1)/2}}{\pi^{(d+1)/2} 2^{(d-1)/2}} \int_{1}^{\infty} \frac{1}{s^{(d+1)/2}\sqrt{s-1}}\, e^{-n|x|^2/(2s)}\, ds \\
&\overset{(*)}{=} \frac{n^{(d-1)/2}}{\pi^{(d+1)/2} 2^{(d-1)/2}}\, B\big(\tfrac{d}{2}, \tfrac{1}{2}\big)\, {}_1F_1\big(\tfrac{d}{2}, \tfrac{d+1}{2}; -\tfrac{n}{2}\, |x|^2\big)
\end{aligned}$$

where $B(\cdot,\cdot)$ is Euler's Beta-function and ${}_1F_1$ is the degenerate hypergeometric function, cf. Gradshteyn–Ryzhik [4, Section 9.20, 9.21] and, for $(*)$, [4, Entry 3.471.5, p. 340].

9 The Variation of Brownian Paths

Problem 9.1 (Solution) Let $\epsilon > 0$ and $\Pi = \{t_0 = 0 < t_1 < \ldots < t_m = 1\}$ be any partition of $[0,1]$. As a continuous function on a compact space, f is uniformly continuous, i.e. there exists $\delta > 0$ such that $|f(x) - f(y)| < \frac{\epsilon}{2m}$ for all $x, y \in [0,1]$ with $|x - y| < \delta$. Pick $n_0 \in \mathbb{N}$ so that $|\Pi_n| < \delta' := \delta \wedge \frac{|\Pi|}{2}$ for all $n \geqslant n_0$.

Now, the balls $\mathbb{B}(t_j, \delta')$ for $0 \leqslant j \leqslant m$ are disjoint as $\delta' \leqslant \frac{|\Pi|}{2}$. Therefore the sets $\mathbb{B}(t_j, \delta') \cap \Pi_{n_0}$ for $0 \leqslant j \leqslant m$ are also disjoint, and non-empty as $|\Pi_{n_0}| < \delta'$. In particular, there exists a subpartition $\Pi' = \{q_0 = 0 < q_1 < \ldots < q_m = 1\}$ of Π_{n_0} such that $|t_j - q_j| < \delta' \leqslant \delta$ for all $0 \leqslant j \leqslant m$. This implies

$$\begin{aligned}
\left| \sum_{j=1}^m |f(t_j) - f(t_{j-1})| - \sum_{j=1}^m |f(q_j) - f(q_{j-1})| \right| &\leqslant \sum_{j=1}^m \big| |f(t_j) - f(t_{j-1})| - |f(q_j) - f(q_{j-1})| \big| \\
&\leqslant \sum_{j=1}^m |f(t_j) - f(q_j) + f(t_{j-1}) - f(q_{j-1})| \\
&\leqslant 2 \cdot \sum_{j=0}^m |f(t_j) - f(q_j)| \\
&\leqslant \epsilon.
\end{aligned}$$

Because adding points to a partition increases the corresponding variation sum, we have

$$S_1^{\Pi}(f,1) \leqslant S_1^{\Pi'}(f,1) + \epsilon \leqslant S_1^{\Pi_{n_0}}(f,1) + \epsilon \leqslant \lim_{n\to\infty} S_1^{\Pi_n}(f,1) + \epsilon \leqslant \mathrm{VAR}_1(f,1) + \epsilon$$

and since Π was arbitrarily chosen, we deduce

$$\mathrm{VAR}_1(f,1) \leqslant \lim_{n\to\infty} S_1^{\Pi_n}(f,1) + \epsilon \leqslant \mathrm{VAR}_1(f,1) + \epsilon$$

for every $\epsilon > 0$. Letting ϵ tend to zero completes the proof.

Remark: The continuity of the function f is essential. A counterexample would be Dirichlet's discontinuous function $f = \mathbb{1}_{\mathbb{Q}\cap[0,1]}$ and Π_n a refining sequence of partitions made up of rational points.

Problem 9.2 (Solution) Note that the problem is straightforward if $\|x\|$ stands for the maximum norm: $\|x\| = \max_{1\leqslant j\leqslant d} |x_j|$.

Remember that all norms on $\mathbb{R}^d$ are equivalent. One quick way of showing this is the following: Denote by e_j with $j \in \{1, \ldots, d\}$ the usual basis of $\mathbb{R}^d$. Then

$$\|x\| \leqslant \left(d \cdot \max_{1\leqslant j\leqslant d} \|e_j\| \right) \cdot \max_{1\leqslant j\leqslant d} |x_j| = \left(d \cdot \max_{1\leqslant j\leqslant d} \|e_j\| \right) \cdot \|x\|_\infty =: B \cdot \|x\|_\infty$$

for every $x = \sum_{j=1}^d x_j e_j$ in $\mathbb{R}^d$ using the triangle inequality and the positive homogeneity of norms. In particular, $x \mapsto \|x\|$ is a continuous mapping from $\mathbb{R}^d$ equipped with the supremum-norm $\|\cdot\|_\infty$ to $\mathbb{R}$, since

$$\big|\|x\| - \|y\|\big| \leqslant \|x-y\| \leqslant B \cdot \|x-y\|_\infty$$

holds for every x, y in $\mathbb{R}^d$. Hence, the extreme value theorem claims that $x \mapsto \|x\|$ attains its minimum on the compact set $\{x \in \mathbb{R}^d : \|x\|_\infty = 1\}$. Finally, this implies $A := \min\{\|x\| : \|x\|_\infty = 1\} > 0$ and hence

$$\|x\| = \left\| \frac{x}{\|x\|_\infty} \right\| \cdot \|x\|_\infty \geqslant A \cdot \|x\|_\infty$$

for every $x \neq 0$ in $\mathbb{R}^d$ as required.

As a result of the equivalence of norms on $\mathbb{R}^d$, it suffices to consider the supremum-norm to determine the finiteness of variations. In particular, $\mathrm{VAR}_p(f;t) < \infty$ if, and only if,

$$\sup\left\{ \sum_{t_{j-1},t_j \in \Pi} |g(t_j) - g(t_{j-1})|^p \vee |h(t_j) - h(t_{j-1})|^p \,:\, \Pi \text{ finite partition of } [0,1] \right\}$$

is finite. But this term is bounded from below by $\mathrm{VAR}_p(g;t) \vee \mathrm{VAR}_p(h;t)$ and from above by $\mathrm{VAR}_p(g;t) + \mathrm{VAR}_p(h;t)$, which proves the desired result.

Problem 9.3 (Solution) Let $p > 0$, $\epsilon > 0$ and $\Pi = \{t_0 = 0 < t_1 < \ldots < t_n = 1\}$ a partition of $[0,1]$. Since f is continuous and the rational numbers are dense in $\mathbb{R}$, there exist $0 < q_1 < \ldots < q_{n-1} < 1$ such that q_j is rational and $|f(t_j) - f(q_j)| < n^{-1/p}\epsilon^{1/p}$ for every $1 \leqslant j \leqslant n-1$. In particular, $\Pi' = \{q_0 = 0 < q_1 < \ldots < q_n = 1\}$ is a rational partition of $[0,1]$ such that $\sum_{j=0}^n |f(t_j) - f(q_j)|^p \leqslant \epsilon$.

Some preliminary considerations: If $\phi : [0,\infty) \to \mathbb{R}$ is concave and $\phi(0) \geqslant 0$ then $\phi(ta) = \phi(ta + (1-t)0) \geqslant t\phi(a) + (1-t)\phi(0) \geqslant t\phi(a)$ for all $a \geqslant 0$ and $t \in [0,1]$. Hence

$$\phi(a+b) = \frac{a}{a+b}\,\phi(a+b) + \frac{b}{a+b}\,\phi(a+b) \leqslant \phi(a) + \phi(b)$$

for all $a, b \geqslant 0$, i.e. ϕ is subadditive. In particular, we have $|x+y|^p \leqslant (|x| + |y|)^p \leqslant |x|^p + |y|^p$ and thus

$$\big||x|^p - |y|^p\big| \leqslant |x-y|^p \quad \text{for all} \quad p \leqslant 1 \quad \text{and} \quad x, y \in \mathbb{R}. \tag{*}$$

For $p > 1$, on the other hand, and $x, y \in \mathbb{R}$ such that $|x| < |y|$ we find

$$\big||y|^p - |x|^p\big| = \int_{|x|}^{|y|} pt^{p-1}\,dt \leqslant p \cdot (|x| \vee |y|)^{p-1} \cdot (|y| - |x|) \leqslant p \cdot (|x| \vee |y|)^{p-1} \cdot |y-x|$$

and hence

$$\big||y|^p - |x|^p\big| \leqslant p \cdot (|x| \vee |y|)^{p-1} \cdot |y-x| \quad \text{for all} \quad p > 1 \quad \text{and} \quad x, y \in \mathbb{R} \tag{**}$$

using the symmetry of the inequality.

Let $p > 0$ and $\epsilon > 0$. For every partition $\Pi = \{t_0 = 0 < t_1 < \ldots < t_n = 1\}$ there exists a rational partition $\Pi' = \{q_0 = 0 < q_1 < \ldots < q_n = 1\}$ such that $\sum_{j=0}^n |f(t_j) - f(q_j)|^{1\wedge p} \leqslant \epsilon$ and hence

$$\begin{aligned}
&\left|\sum_{j=1}^n |f(t_j) - f(t_{j-1})|^p - \sum_{j=1}^n |f(q_j) - f(q_{j-1})|^p\right| \\
&\leqslant \sum_{j=1}^n \left||f(t_j) - f(t_{j-1})|^p - |f(q_j) - f(q_{j-1})|^p\right| \\
&\underset{(**)}{\overset{(*)}{\leqslant}} \max\left\{1,\ (p\cdot 2^{p-1}\cdot \|f\|_\infty^{p-1})\right\}\cdot \sum_{j=1}^n |f(t_j) - f(q_j) + f(t_{j-1}) - f(q_{j-1})|^{1\wedge p} \\
&\leqslant C\cdot \sum_{j=0}^n |f(t_j) - f(q_j)|^{1\wedge p} \\
&\leqslant C\cdot\epsilon
\end{aligned}$$

with a finite constant $C > 0$.

In particular, we have $\mathrm{VAR}_p(f;1) - C\cdot\epsilon \leqslant \mathrm{VAR}_p^{\mathbb{Q}}(f;1) \leqslant \mathrm{VAR}_p(f;1)$ where

$$\mathrm{VAR}_p^{\mathbb{Q}}(f;1) := \sup\left\{\sum_{q_{j-1},q_j\in\Pi'} |f(q_j) - f(q_{j-1})|^p \ :\ \Pi' \text{ finite, rational partition of } [0,1]\right\}$$

and hence the desired result as ϵ tends to zero.

<u>Alternative Approach:</u> Note that $(\xi_1,\ldots,\xi_n) \mapsto \sum_{j=1}^n |f(\xi_j) - f(\xi_{j-1})|^p$ is a continuous map since it is the finite sum and composition of continuous maps, and that the rational numbers are dense in $\mathbb{R}$.

Problem 9.4 (Solution) Obviously, we have $\mathrm{VAR}_p^\circ(f;t) \leqslant \mathrm{VAR}_p(f;t)$ with

$$\mathrm{VAR}_p^\circ(f;t) := \sup\left\{\sum_{j=1}^n |f(s_j) - f(s_{j-1})|^p \ :\ n\in\mathbb{N} \text{ and } 0 < s_0 < s_1 < \ldots < s_n < t\right\}$$

because there are less (non-negative) summands in the definition of $\mathrm{VAR}_p^\circ(f;t)$.

Let $\epsilon > 0$ and $\Pi = \{t_0 = 0 < t_1 < \ldots < t_n = t\}$ a partition of $[0,t]$. Set $s_j = t_j$ for $1 \leqslant j \leqslant n-1$ and note that $\xi \mapsto |f(\xi_0) - f(\xi)|^p$ is a continuous map for every $\xi_0 \in [0,t]$ since it is the composition of continuous maps. Hence we can pick $s_0 \in (t_0,t_1)$ and $s_n \in (t_{n-1},t_n)$ with

$$\begin{aligned}
\left||f(s_1) - f(t_0)|^p - |f(s_1) - f(s_0)|^p\right| &< \frac{\varepsilon}{2} \\
\left||f(t_n) - f(t_{n-1})|^p - |f(s_n) - f(t_{n-1})|^p\right| &< \frac{\varepsilon}{2}
\end{aligned}$$

and so that $0 < s_0 < s_1 < \ldots < s_n < t$. This implies

$$\begin{aligned}
\sum_{j=1}^n |f(t_j) - f(t_{j-1})|^p &= |f(s_1) - f(t_0)|^p + \sum_{j=2}^{n-1} |f(s_j) - f(s_{j-1})|^p + |f(t_n) - f(s_{n-1})|^p \\
&\leqslant \frac{\varepsilon}{2} + \sum_{j=1}^n |f(s_j) - f(s_{j-1})|^p + \frac{\varepsilon}{2}
\end{aligned}$$

$$\leqslant \epsilon + \mathrm{VAR}_p^\circ(f;t)$$

and thus $\mathrm{VAR}_p(f;t) \leqslant \epsilon + \mathrm{VAR}_p^\circ(f;t)$ since the partition $\Pi = \{t_0 = 0 < t_1 < \ldots < t_n = t\}$ was arbitrarily chosen. Consequently, $\mathrm{VAR}_p(f;t) \leqslant \mathrm{VAR}_p^\circ(f;t)$ as ϵ tends to zero, as required.

The same argument shows that $\mathrm{var}_p(f;t)$ does not change its value (if it exists).

Problem 9.5 (Solution) a) Use $B(t) - B(s) \sim \mathsf{N}(0, |t-s|)$ to find

$$\begin{aligned}
\mathbb{E}\,Y_n &= \sum_{k=1}^n \mathbb{E}\left(B\left(\tfrac{k}{n}\right) - B\left(\tfrac{k-1}{n}\right)\right)^2\\
&= \sum_{k=1}^n \mathbb{V}\left(B\left(\tfrac{k}{n}\right) - B\left(\tfrac{k-1}{n}\right)\right)\\
&= \sum_{k=1}^n \left(\tfrac{k}{n} - \tfrac{k-1}{n}\right)\\
&= \sum_{k=1}^n \tfrac{1}{n}\\
&= 1
\end{aligned}$$

and the independence of increments to get

$$\begin{aligned}
\mathbb{V}\,Y_n &= \sum_{k=1}^n \mathbb{V}\left(B\left(\tfrac{k}{n}\right) - B\left(\tfrac{k-1}{n}\right)\right)^2\\
&= \sum_{k=1}^n \mathbb{E}\left(B\left(\tfrac{k}{n}\right) - B\left(\tfrac{k-1}{n}\right)\right)^4 - \left(\mathbb{E}\left(B\left(\tfrac{k}{n}\right) - B\left(\tfrac{k-1}{n}\right)\right)^2\right)^2\\
&= \sum_{k=1}^n 3\cdot\left(\tfrac{k}{n} - \tfrac{k-1}{n}\right)^2 - \left(\tfrac{k}{n} - \tfrac{k-1}{n}\right)^2\\
&= 2\cdot\sum_{k=1}^n \tfrac{1}{n^2}\\
&= 2\cdot\tfrac{1}{n}
\end{aligned}$$

where we also used that $\mathbb{E}(X^4) = 3\cdot\sigma^4$ for $X \sim \mathsf{N}(0,\sigma^2)$.

b) Note that the increments $B(\frac{k}{n}) - B(\frac{k-1}{n}) \sim \mathsf{N}(0, \frac{1}{n})$ are iid random variables. By a standard result the sum of squares $n\sum_{k=1}^n \left(B(\frac{k}{n}) - B(\frac{k-1}{n})\right)^2$ has a χ^2_n-distribution, i.e. its density is given by

$$2^{-n/2}\,\frac{1}{\Gamma\left(\frac{n}{2}\right)}\,s^{\frac{n}{2}-1}\,e^{-\frac{s}{2}}\,\mathbb{1}_{[0,\infty)}(s).$$

and we get

$$\sum_{k=1}^n \left(B(\tfrac{k}{n}) - B(\tfrac{k-1}{n})\right)^2 \sim 2^{-n/2}\,\frac{n}{\Gamma\left(\frac{n}{2}\right)}\,(ns)^{\frac{n}{2}-1}\,e^{-\frac{ns}{2}}\,\mathbb{1}_{[0,\infty)}(s).$$

<u>Here is the calculation:</u> (in case you do not know this standard result...): If $X \sim \mathsf{N}(0,1)$ and $x > 0$, we have

$$\mathbb{P}(X^2 \leqslant x) = \mathbb{P}(X \leqslant \sqrt{x}) = \frac{1}{\sqrt{2\pi}}\int_{-\sqrt{x}}^{\sqrt{x}} \exp\left(-\frac{t^2}{2}\right)dt$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\sqrt{x}} \exp\left(-\frac{t^2}{2}\right) dt$$
$$= \frac{1}{\sqrt{2\pi}} \int_0^{x} \exp\left(-\frac{s}{2}\right) \cdot s^{-1/2} ds$$

using the change of variable $s = t^2$. Hence, X^2 has density

$$f_{X^2}(s) = \mathbb{1}_{(0,\infty)}(s) \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{s}{2}\right) \cdot s^{-1/2}.$$

Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with $X_1 \sim \mathsf{N}(0,1)$. We want to prove by induction that for $n \geqslant 1$

$$f_{X_1^2+\ldots+X_n^2}(s) = C_n \cdot \mathbb{1}_{(0,\infty)}(s) \cdot \exp\left(-\frac{s}{2}\right) \cdot s^{n/2-1}$$

with some normalizing constants $C_n > 0$. Assume that this is true for $1, \ldots, n$. Since X_{n+1}^2 is independent of $X_1^2 + \ldots + X_n^2$ and distributed like X_1^2, we know that the density of the sum is a convolution. This leads to

$$\begin{aligned}
f_{X_1^2+\ldots+X_{n+1}^2}(s) &= \int_{-\infty}^{\infty} f_{X_1^2+\ldots+X_n^2}(t) \cdot f_{X_{n+1}^2}(s-t)\, dt \\
&= C_n \cdot C_1 \cdot \int_0^s \exp\left(-\frac{t}{2}\right) \cdot t^{n/2-1} \cdot \exp\left(-\frac{s-t}{2}\right) \cdot (s-t)^{-1/2}\, dt \\
&= C_n \cdot C_1 \cdot \exp\left(-\frac{s}{2}\right) \cdot \int_0^s t^{n/2-1} \cdot (s-t)^{-1/2}\, mdt \\
&= C_n \cdot C_1 \cdot \exp\left(-\frac{s}{2}\right) \cdot s^{n/2-1} \cdot s^{-1/2} \cdot \int_0^s \left(\frac{t}{s}\right)^{n/2-1} \cdot \left(1 - \frac{t}{s}\right)^{-1/2} dt \\
&= C_n \cdot C_1 \cdot \exp\left(-\frac{s}{2}\right) \cdot s^{(n+1)/2-1} \cdot \int_0^1 x^{n/2-1} \cdot (1-x)^{-1/2}\, dx \\
&= C_{n+1} \cdot \exp\left(-\frac{s}{2}\right) \cdot s^{(n+1)/2-1}
\end{aligned}$$

using the change of variable $x = t/s$. Since probability distribution functions integrate to one, we find

$$\begin{aligned}
1 = C_n \cdot \int_0^{\infty} \exp\left(-\frac{s}{2}\right) \cdot s^{n/2-1}\, ds &= C_n \cdot 2^{n/2} \int_0^{\infty} \exp(-t) \cdot t^{n/2-1}\, dt \\
&= C_n \cdot 2^{n/2} \cdot \Gamma(n/2)
\end{aligned}$$

and thus

$$f_{X_1^2+\ldots+X_n^2}(s) = \left(2^{n/2} \cdot \Gamma(n/2)\right)^{-1} \cdot \mathbb{1}_{(0,\infty)}(s) \cdot e^{-s/2} \cdot s^{n/2-1}$$

which is usually called *chi-squared or χ^2-distribution with n degrees of freedom.* Now, remember that $B\left(\frac{k}{n}\right) - B\left(\frac{k-1}{n}\right) \sim \mathsf{N}(0, 1/n) \sim n^{-1/2} \cdot X_k$ for $1 \leqslant k \leqslant n$. Hence

$$\begin{aligned}
f_{Y_n}(s) &= n \cdot f_{X_1^2+\ldots+X_n^2}(n \cdot s) \\
&= n \cdot \left(2^{n/2} \cdot \Gamma(n/2)\right)^{-1} \cdot \mathbb{1}_{(0,\infty)}(s) \cdot e^{-n \cdot s/2} \cdot (ns)^{n/2-1}.
\end{aligned}$$

c) For $X \in \mathsf{N}(0,1)$ and $\xi < 1/2$, we find

$$\mathbb{E}(e^{\xi \cdot X^2}) = (2 \cdot \pi)^{-1/2} \int_{-\infty}^{\infty} e^{\xi \cdot x^2} e^{-x^2/2}\, dx = \frac{2}{\sqrt{2 \cdot \pi}} \int_0^{\infty} e^{-1/2 \cdot (1-2\xi) \cdot x^2}\, dx$$

$$= (1-2\xi)^{-1/2}\frac{2}{\sqrt{2\cdot\pi}}\int_0^\infty e^{-y^2/2}\,dy$$
$$= (1-2\xi)^{-1/2}$$

using the change of variable $x^2 = (1-2\xi)y^2$. Since the moment generating function $\xi \mapsto (1-2\xi)^{-1/2}$ has a unique analytic extension to an open strip around the imaginary axis, the characteristic function is of the form

$$\mathbb{E}(e^{i\cdot\xi\cdot X^2}) = (1-2i\xi)^{-1/2}.$$

Using the independence and $B\left(\frac{k}{n}\right) - B\left(\frac{k-1}{n}\right) \sim \mathsf{N}(0, 1/n)$, we obtain

$$\mathbb{E}(e^{i\cdot\xi\cdot Y_n}) = \prod_{k=1}^n \mathbb{E}(e^{i\cdot\xi\cdot(B_{k/n}-B_{(k-1)/n})^2}) = \prod_{k=1}^n \mathbb{E}(e^{i\cdot(\xi/n)\cdot X^2}) = (1-2i(\xi/n))^{-n/2}$$

and hence

$$\lim_{n\to\infty}\phi_n(\xi) = \lim_{n\to\infty}(1-2i(\xi/n))^{-n/2} = \left(\lim_{n\to\infty}\left(1-\frac{2i\xi}{n}\right)^n\right)^{-1/2} = \left(e^{-2i\xi}\right)^{-1/2} = e^{i\xi}.$$

(d) We have shown in a) that $\mathbb{E}\left((Y_n-1)^2\right) = \mathbb{V}(Y_n) = 2/n$ which tends to zero as $n\to\infty$.

Problem 9.6 (Solution) (a)

$$\sqrt{2\pi}\cdot\mathbb{P}(Z>x) = \int_x^\infty e^{-y^2/2}dy > \int_x^\infty \frac{y}{x}\cdot e^{-y^2/2}dy = \frac{1}{x}\cdot\left[-e^{-y^2/2}\right]_x^\infty = \frac{1}{x}\cdot e^{-x^2/2}$$
$$\implies \mathbb{P}(Z>x) < \frac{1}{\sqrt{2\pi}}\frac{e^{-x^2/2}}{x}$$

On the other hand

$$\begin{aligned}
\sqrt{2\pi}\cdot\mathbb{P}(Z>x) &= \int_x^\infty e^{-y^2/2}dy\\
&< \int_x^\infty \frac{x^2}{y^2}\cdot e^{-y^2/2}dy\\
&= x^2\cdot\left(\left[-\frac{1}{y}\cdot e^{-y^2/2}\right]_x^\infty - \int_x^\infty e^{-y^2/2}dy\right)\\
&= x^2\cdot\left(\left[-\frac{1}{y}\cdot e^{-y^2/2}\right]_x^\infty - \sqrt{2\pi}\cdot\mathbb{P}(Z>x)\right)\\
&\implies (1+x^2)\cdot\sqrt{2\pi}\cdot\mathbb{P}(Z>x) \geqslant x\cdot e^{-x^2/2}\\
&\implies \mathbb{P}(Z>x) > \frac{1}{\sqrt{2\pi}}\frac{xe^{-x^2/2}}{x^2+1}
\end{aligned}$$

(b) Using the independence of $A_{k,n}$ for $1\leqslant k\leqslant 2^n$, we find

$$\begin{aligned}
\mathbb{P}\left(\varlimsup_{n\to\infty}\bigcup_{k=1}^{2^n}A_{k,n}\right) &= 1-\mathbb{P}\left(\liminf_{n\to\infty}\bigcap_{k=1}^{2^n}A_{k,n}^c\right)\\
&\geqslant 1-\liminf_{n\to\infty}\mathbb{P}\left(\bigcap_{k=1}^{2^n}A_{k,n}^c\right)
\end{aligned}$$

$$= 1 - \liminf_{n\to\infty} \prod_{k=1}^{2^n} \mathbb{P}(A_{k,n}^c)$$

and hence it suffices to prove $\liminf_{n\to\infty} \prod_{k=1}^{2^n} \mathbb{P}(A_{k,n}^c) = 0$.

Since $1 - x \leqslant e^{-x}$ for $x \geqslant 0$, we obtain

$$\prod_{k=1}^{2^n} \mathbb{P}(A_{k,n}^c) = \Big(1 - \mathbb{P}(A_{1,n})\Big)^{2^n} \leqslant e^{-2^n \cdot \mathbb{P}(A_{1,n})}$$

and a) implies

$$\begin{aligned} 2^n \cdot \mathbb{P}(A_{1,n}) &= 2^n \cdot \mathbb{P}(\sqrt{2^{-n}} \cdot |Z| > c\sqrt{n2^{-n}}) \\ &= 2^{n+1} \cdot \mathbb{P}(Z > c\sqrt{n}) \\ &\geqslant \frac{2^{n+1}}{\sqrt{2\pi}} \cdot \frac{c\sqrt{n}}{c^2 n + 1} \cdot e^{-c^2 n/2}. \end{aligned}$$

Now, $(c^2 n)/(c^2 n + 1) \to 1$ as $n \to \infty$ and thus there exists some $n_0 \in \mathbb{N}$ such that

$$\frac{c^2 n}{c^2 n + 1} \geqslant \frac{1}{2} \iff \frac{c\sqrt{n}}{c^2 n + 1} \geqslant \frac{1}{2c\sqrt{n}}$$

for all $n \geqslant n_0$. Therefore, we have

$$2^n \cdot \mathbb{P}(A_{1,n}) \geqslant \frac{2^n}{\sqrt{2\pi}} \cdot \frac{1}{c\sqrt{n}} \cdot e^{-c^2 n/2} = \frac{1}{\sqrt{2\pi}c} \cdot \frac{1}{\sqrt{n}} \cdot e^{(\log(2) - c^2/2)n}$$

for $n \geqslant n_0$. Since $\ln(2) - c^2/2 > 0$ if, and only if, $c < \sqrt{2\log(2)}$, we have $2^n \cdot \mathbb{P}(A_{1,n}) \to \infty$ and thus $\liminf_{n\to\infty} \prod_{k=1}^{2^n} \mathbb{P}(A_{k,n}^C) = 0$ if $c < \sqrt{2\log(2)}$.

c) With $c < \sqrt{2\log(2)}$ we deduce

$$\begin{aligned} 1 &= \mathbb{P}\left(\limsup_{n\to\infty} \bigcup_{k=1}^{2^n} A_{k,n}\right) \\ &= \mathbb{P}\Big(\big\{\omega \in \Omega : \text{ for infinitely many } n \in \mathbb{N} \;\; \exists k \in \{1, \dots, 2^n\} \\ &\qquad\qquad \text{such that } |B(k2^{-n})(\omega) - B((k-1)2^{-n})(\omega)| > c\sqrt{n\, 2^{-n}}\big\}\Big) \\ &= \mathbb{P}\Big(\big\{\omega \in \Omega : \text{ for infinitely many } n \in \mathbb{N} \;\; \exists k \in \{1, \dots, 2^n\} \\ &\qquad\qquad \text{such that } \frac{|B(k2^{-n})(\omega) - B((k-1)2^{-n})(\omega)|}{\sqrt{2^{-n}}} > c\sqrt{n}\big\}\Big) \\ &\leqslant \mathbb{P}\Big(\big\{\omega \in \Omega : t \mapsto B_t(\omega) \;\text{ is NOT 1/2-Hölder continuous}\big\}\Big). \end{aligned}$$

Problem 9.7 (Solution) From Problem 9.5 we know that

$$\Phi(\lambda) = \mathbb{E}(e^{\lambda(X^2-1)}) = e^{-\lambda}\,\mathbb{E}(e^{\lambda X^2}) = e^{-\lambda}(1 - 2\lambda)^{-1/2} \quad \text{for all } \; 0 < \lambda < 1/2.$$

Using $(a-b)^2 \leqslant 2\,(a^2 + b^2)$, we get

$$|(X^2-1)^2 e^{\lambda(X^2-1)}| \leqslant |X^2 - 1|^2 \cdot e^{\lambda X^2} \leqslant 2(X^4 + 1) \cdot e^{\lambda_0 X^2}.$$

Since $\lambda < \lambda_0 < 1/2$ there is some $\epsilon > 0$ such that $\lambda < \lambda_0 < \lambda_0 + \epsilon < 1/2$. Thus,

$$|(X^2-1)^2 e^{\lambda(X^2-1)}| \leqslant 2(X^4+1)e^{-\epsilon X^2} \cdot e^{(\lambda_0+\epsilon)X^2}.$$

It is straightforward to see that

$$2(X^4+1)e^{-\epsilon X^2} \leqslant C_\epsilon = C(\lambda_0) < \infty,$$

and the claim follows.

Problem 9.8 (Solution) Using the notation

$$L(n) := \left(\sum_{j=1}^n a_j\right)^4 + 3\cdot\sum_{j=1}^n a_j^4 - 4\cdot\left(\sum_{j=1}^n a_j^3\right)\cdot\left(\sum_{k=1}^n a_k\right) + 2\sum_{j=1}^n\sum_{k=j+1}^n a_j^2 a_k^2$$

$$R(n) := 2\cdot\sum_{j=1}^n a_j^2\Bigg(\sum_{\substack{k=1\\k\neq j}}^n a_k\Bigg)^2 + 4\cdot\left(\sum_{j=1}^n\sum_{k=j+1}^n a_j a_k\right)^2$$

we deduce:

a) Start of the induction:

$$\begin{aligned}
L(2) &= (a_1+a_2)^4 + 3(a_1^4+a_2^4) - 4(a_1^3+a_2^3)(a_1+a_2) + 2a_1^2a_2^2\\
&= (a_1^4 + 4a_1^3a_2 + 6a_1^2a_2^2 + 4a_1a_2^3 + a_2^4) + 3(a_1^4+a_2^4)\\
&\quad - 4(a_1^4 + a_2^4 + a_1^3a_2 + a_2^3a_1) + 2a_1^2a_2^2\\
&= 6a_1^2a_2^2 + 2a_1^2a_2^2\\
&= 2(a_1^2a_2^2 + a_2^2a_1^2) + 4a_1^2a_2^2\\
&= R(2)
\end{aligned}$$

b) Induction step: Assume that we have already shown that the statement is true for n. Then

$$\begin{aligned}
L(n+1) &= \left(\sum_{j=1}^n a_j + a_{n+1}\right)^4 + 3\cdot\left(\sum_{j=1}^n a_j^4 + a_{n+1}^4\right) - 4\cdot\left(\sum_{j=1}^n a_j^3 + a_{n+1}^3\right)\cdot\left(\sum_{k=1}^n a_k + a_{n+1}\right)\\
&\quad + 2\left(\sum_{j=1}^n\sum_{k=j+1}^n a_j^2a_k^2 + \sum_{j=1}^n a_j^2 a_{n+1}^2\right)\\
&= L(n) + 4\left(\sum_{j=1}^n a_j\right)^3 a_{n+1} + 6\left(\sum_{j=1}^n a_j\right)^2 a_{n+1}^2 + 4\left(\sum_{j=1}^n a_j\right)a_{n+1}^3 + a_{n+1}^4\\
&\quad + 3a_{n+1}^4 - 4a_{n+1}^4 - 4\left(\sum_{j=1}^n a_j^3\right)a_{n+1} - 4a_{n+1}^3\left(\sum_{j=1}^n a_j\right) + 2\sum_{j=1}^n a_j^2 a_{n+1}^2\\
&= L(n) + 4\left(\sum_{j=1}^n a_j\right)^3 a_{n+1} + 6\left(\sum_{j=1}^n a_j\right)^2 a_{n+1}^2 - 4\left(\sum_{j=1}^n a_j^3\right)a_{n+1} + 2\sum_{j=1}^n a_j^2 a_{n+1}^2\\
&= L(n) + 4a_{n+1}\left(\sum_{j=1}^n a_j\right)^3 + 6a_{n+1}^2\left(\sum_{j=1}^n a_j\right)^2 - 4a_{n+1}\left(\sum_{j=1}^n a_j^3\right) + 2a_{n+1}^2\sum_{j=1}^n a_j^2
\end{aligned}$$

$$\begin{aligned}
R(n+1) &= 2\cdot\sum_{j=1}^{n+1} a_j^2\Bigg(\sum_{\substack{k=1\\k\neq j}}^{n+1} a_k\Bigg)^2 + 4\cdot\Bigg(\sum_{j=1}^{n+1}\sum_{k=j+1}^{n+1} a_j a_k\Bigg)^2\\
&= R(n) + 2a_{n+1}^2\Bigg(\sum_{k=1}^{n} a_k\Bigg)^2 + 2\sum_{j=1}^{n} a_j^2\Bigg(a_{n+1}^2 + 2a_{n+1}\sum_{\substack{k=1\\k\neq j}}^{n} a_k\Bigg) + 4\Bigg(\sum_{j=1}^{n} a_j a_{n+1}\Bigg)^2\\
&\quad + 4\cdot 2\cdot\Bigg(\sum_{j=1}^{n}\sum_{k=j+1}^{n} a_j a_k\Bigg)\Bigg(\sum_{i=1}^{n} a_i a_{n+1}\Bigg)\\
&= R(n) + 2a_{n+1}^2\Bigg(\sum_{k=1}^{n} a_k\Bigg)^2 + 2a_{n+1}^2\sum_{j=1}^{n} a_j^2 + 4a_{n+1}\sum_{j=1}^{n}\sum_{\substack{k=1\\k\neq j}}^{n} a_j^2 a_k + 4a_{n+1}^2\Bigg(\sum_{j=1}^{n} a_j\Bigg)^2\\
&\quad + 4a_{n+1}\cdot\Bigg(\sum_{j=1}^{n}\sum_{\substack{k=1\\k\neq j}}^{n} a_j a_k\Bigg)\Bigg(\sum_{i=1}^{n} a_i\Bigg)\\
&= R(n) + 6a_{n+1}^2\Bigg(\sum_{k=1}^{n} a_k\Bigg)^2 + 2a_{n+1}^2\sum_{j=1}^{n} a_j^2 + 4a_{n+1}\sum_{j=1}^{n}\sum_{\substack{k=1\\k\neq j}}^{n} a_j^2 a_k\\
&\quad + 4a_{n+1}\cdot\Bigg(\sum_{j=1}^{n}\sum_{\substack{k=1\\k\neq j}}^{n} a_j a_k\Bigg)\Bigg(\sum_{i=1}^{n} a_i\Bigg)
\end{aligned}$$

and hence $L(n+1) = R(n+1)$ if, and only if,

$$4a_{n+1}\Bigg(\sum_{j=1}^{n} a_j\Bigg)^3 - 4a_{n+1}\Bigg(\sum_{j=1}^{n} a_j^3\Bigg) = 4a_{n+1}\sum_{j=1}^{n}\sum_{\substack{k=1\\k\neq j}}^{n} a_j^2 a_k + 4a_{n+1}\cdot\Bigg(\sum_{j=1}^{n}\sum_{\substack{k=1\\k\neq j}}^{n} a_j a_k\Bigg)\Bigg(\sum_{i=1}^{n} a_i\Bigg)$$

if, and only if, $a_{n+1} = 0$ or $a_{n+1} \neq 0$ and

$$\Bigg(\sum_{j=1}^{n} a_j\Bigg)^3 - \Bigg(\sum_{j=1}^{n} a_j^3\Bigg) = \sum_{j=1}^{n}\sum_{\substack{k=1\\k\neq j}}^{n} a_j^2 a_k + \Bigg(\sum_{j=1}^{n}\sum_{\substack{k=1\\k\neq j}}^{n} a_j a_k\Bigg)\Bigg(\sum_{i=1}^{n} a_i\Bigg)$$

But the second term on the right hand side is

$$\begin{aligned}
&\Bigg(\sum_{j=1}^{n}\sum_{\substack{k=1\\k\neq j}}^{n} a_j a_k\Bigg)\Bigg(\sum_{i=1}^{n} a_i\Bigg)\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\substack{k=1\\k\neq j}}^{n} a_i a_j a_k\\
&= \sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}\sum_{\substack{k=1\\k\neq j\\k\neq i}}^{n} a_i a_j a_k + \sum_{i=j=1}^{n}\sum_{\substack{k=1\\k\neq j}}^{n} a_i^2 a_k + \sum_{i=k=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n} a_i^2 a_j
\end{aligned}$$

and hence $L(n+1) = R(n+1)$ if, and only if, $a_{n+1} = 0$ or $a_{n+1} \neq 0$ and

$$\begin{aligned}
&\Bigg(\sum_{j=1}^{n} a_j\Bigg)^3 - \Bigg(\sum_{j=1}^{n} a_j^3\Bigg) = 3\sum_{j=1}^{n}\sum_{\substack{k=1\\k\neq j}}^{n} a_j^2 a_k + \sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}\sum_{\substack{k=1\\k\neq j\\k\neq i}}^{n} a_i a_j a_k\\
\iff &\Bigg(\sum_{j=1}^{n} a_j\Bigg)^3 = \Bigg(\sum_{j=1}^{n} a_j^3\Bigg) + 3\sum_{j=1}^{n}\sum_{\substack{k=1\\k\neq j}}^{n} a_j^2 a_k + \sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}\sum_{\substack{k=1\\k\neq j\\k\neq i}}^{n} a_i a_j a_k
\end{aligned}$$

which is obviously true.

Problem 9.9 (Solution) We prove this statement inductively. The statement is obviously true for $n = 1$. Assume that it holds for n. Then

$$\begin{aligned}
\left|\prod_{j=1}^{n+1} a_j - \prod_{j=1}^{n+1} b_j\right| &= \left|\prod_{j=1}^{n+1} a_j - \left(\prod_{j=1}^{n} b_j\right)\cdot a_{n+1} + \left(\prod_{j=1}^{n} b_j\right)\cdot a_{n+1} - \prod_{j=1}^{n+1} b_j\right| \\
&\leqslant \left|\prod_{j=1}^{n} a_j - \prod_{j=1}^{n} b_j\right| \cdot \left|a_{n+1}\right| + \left|\prod_{j=1}^{n} b_j\right| \cdot \left|a_{n+1} - b_{n+1}\right| \\
&\leqslant \sum_{j=1}^{n} |a_j - b_j| + |a_{n+1} - b_{n+1}| \\
&= \sum_{j=1}^{n+1} |a_j - b_j|
\end{aligned}$$

as required.

10 Regularity of Brownian Paths

Problem 10.1 (Solution) a) Note that for $t, h \geqslant 0$ and any integer $k = 0, 1, 2, \dots$

$$\mathbb{P}(N_{t+h} - N_t = k) = \mathbb{P}(N_h = k) = \frac{(\lambda h)^k}{k!} e^{-\lambda h}.$$

This shows that we have for any $\alpha > 0$

$$\begin{aligned}
\mathbb{E}\left(|N_{t+h} - N_t|^\alpha\right) &= \sum_{k=0}^\infty k^\alpha \frac{(\lambda h)^k}{k!} e^{-\lambda h} \\
&= \lambda h\, e^{-\lambda h} + \sum_{k=2}^\infty k^\alpha \frac{(\lambda h)^k}{k!} e^{-\lambda h} \\
&= \lambda h\, e^{-\lambda h} + \lambda h \sum_{k=2}^\infty k^\alpha \frac{(\lambda h)^{k-1}}{k!} e^{-\lambda h} \\
&= \lambda h\, e^{-\lambda h} + \mathsf{o}(h)
\end{aligned}$$

and, thus,

$$\lim_{h\to 0} \frac{\mathbb{E}\left(|N_{t+h} - N_t|^\alpha\right)}{h} = \lambda$$

which means that (10.1) cannot hold for any $\alpha > 0$ and $\beta > 0$.

b) Part a) shows also $\mathbb{E}\left(|N_{t+h} - N_t|^\alpha\right) \leqslant c\, h$, i.e. condition (10.1) holds for $\alpha > 0$ and $\beta = 0$.

The fact that $\beta = 0$ is needed for the convergence of the dyadic series (with the power $\gamma < \beta/\alpha$) in the proof of Theorem 10.1.

c) We have

$$\begin{aligned}
\mathbb{E}(N_t) &= \sum_{k=0}^\infty k \frac{t^k}{k!} e^{-t} = \sum_{k=1}^\infty k \frac{t^k}{k!} e^{-t} = t \sum_{k=1}^\infty \frac{t^{k-1}}{(k-1)!} e^{-t} = t \sum_{j=0}^\infty \frac{t^j}{j!} e^{-t} = t \\
\mathbb{E}(N_t^2) &= \sum_{k=0}^\infty k^2 \frac{t^k}{k!} e^{-t} = \sum_{k=1}^\infty k^2 \frac{t^k}{k!} e^{-t} = t \sum_{k=1}^\infty k \frac{t^{k-1}}{(k-1)!} e^{-t} \\
&= t \sum_{k=1}^\infty (k-1) \frac{t^{k-1}}{(k-1)!} e^{-t} + t \sum_{k=1}^\infty \frac{t^{k-1}}{(k-1)!} e^{-t} \\
&= t^2 \sum_{k=2}^\infty \frac{t^{k-2}}{(k-2)!} e^{-t} + t \sum_{k=1}^\infty \frac{t^{k-1}}{(k-1)!} e^{-t} = t^2 + t
\end{aligned}$$

and this shows that

$$\begin{aligned}
\mathbb{E}(N_t - t) &= \mathbb{E}\, N_t - t = 0 \\
\mathbb{E}\left((N_t - t)^2\right) &= \mathbb{E}(N_t^2) - 2t\, \mathbb{E}\, N_t + t^2 = t
\end{aligned}$$

and, finally, if $s \leqslant t$

$$\begin{aligned}
\operatorname{Cov}\big((N_t - t)(N_s - s)\big) &= \mathbb{E}\big((N_t - t)(N_s - s)\big)\\
&= \mathbb{E}\big((N_t - N_s - t + s)(N_s - s)\big) + \mathbb{E}\big((N_s - s)^2\big)\\
&= \mathbb{E}\big((N_t - N_s - t + s)\big)\,\mathbb{E}\big((N_s - s)\big) + s\\
&= s = s \wedge t
\end{aligned}$$

where we used the independence of $N_t - N_s \perp\!\!\!\perp N_s$.

Alternative Solution: One can show, as for a Brownian motion (Example 5.2 a)), that N_t is a martingale for the canonical filtration $\mathcal{F}_t^N = \sigma(N_s : s \leqslant t)$. The proof only uses stationary and independent increments. Thus, by the tower property, pull out and the martingale property,

$$\begin{aligned}
\mathbb{E}\big((N_t - t)(N_s - s)\big) &= \mathbb{E}\big(\,\mathbb{E}\big((N_t - t)(N_s - s)\,\big|\,\mathcal{F}_s^N\big)\big)\\
&= \mathbb{E}\big((N_s - s)\,\mathbb{E}\big((N_t - t)\,\big|\,\mathcal{F}_s^N\big)\big)\\
&= \mathbb{E}\big((N_s - s)^2\big)\\
&= s = s \wedge t.
\end{aligned}$$

Problem 10.2 (Solution) We have

$$\max_{1\leqslant j\leqslant n} |x_j|^p \leqslant \max_{1\leqslant j\leqslant n}\Big(|x_1|^p + \dots + |x_n|^p\Big) = \sum_{j=1}^n |x_j|^p \leqslant \sum_{j=1}^n \max_{1\leqslant k\leqslant n} |x_k|^p = n \max_{1\leqslant k\leqslant n} |x_k|^p.$$

Since $\max_{1\leqslant j\leqslant n} |x_j|^p = \big(\max_{1\leqslant j\leqslant n} |x_j|\big)^p$ the claim follows (actually with $n^{1/p}$ which is smaller than n....)

Problem 10.3 (Solution) Let $\alpha \in (0,1)$. Since

$$|x+y|^\alpha \leqslant (|x| + |y|)^\alpha$$

it is enough to show that

$$(|x| + |y|)^\alpha \leqslant |x|^\alpha + |y|^\alpha$$

and, without loss of generality

$$(s+t)^\alpha \leqslant s^\alpha + t^\alpha \quad \forall s,t > 0.$$

This follows from

$$s^\alpha + t^\alpha = s\cdot s^{\alpha-1} + t\cdot t^{\alpha-1} \geqslant s\cdot(s+t)^{\alpha-1} + t\cdot(s+t)^{\alpha-1} = (s+t)(s+t)^{\alpha-1} = (s+t)^\alpha.$$

Since the expectation is linear, this proves that

$$\mathbb{E}(|X+Y|^\alpha) \leqslant \mathbb{E}(|X|^\alpha) + \mathbb{E}(|Y|^\alpha).$$

In the proof of Theorem 10.1 (page 154, line 1 from above and onwards) we get:

This entails for $\alpha \in (0,1)$ because of the subadditivity of $x \mapsto |x|^\alpha$

$$\begin{aligned}
\left(\sup_{x,y\in D,\, x\neq y} \frac{|\xi(x)-\xi(y)|}{|x-y|^\gamma}\right)^\alpha &= \sup_{m\geqslant 0} \sup_{\substack{x,y\in D\\ 2^{-m-1}\leqslant |x-y|<2^{-m}}} \frac{|\xi(x)-\xi(y)|^\alpha}{2^{-(m+1)\gamma\alpha}}\\
&\leqslant \sup_{m\geqslant 0}\left(2^\alpha \cdot 2^{(m+1)\gamma\alpha} \sum_{j\geqslant m} \sigma_j^\alpha\right)\\
&= 2^{(1+\gamma)\alpha} \sup_{m\geqslant 0} \sum_{j\geqslant m} 2^{m\gamma\alpha}\sigma_j^\alpha\\
&\leqslant 2^{(1+\gamma)\alpha} \sum_{j=0}^\infty 2^{j\gamma\alpha}\sigma_j^\alpha.
\end{aligned}$$

For $\alpha \in (0,1)$ and $\alpha\gamma < \beta$ we get

$$\begin{aligned}
\mathbb{E}\left[\left(\sup_{x\neq y,\, x,y\in D} \frac{|\xi(x)-\xi(y)|}{|x-y|^\gamma}\right)^\alpha\right] &\leqslant 2^{(1+\gamma)\alpha} \sum_{j=0}^\infty 2^{j\gamma\alpha}\, \mathbb{E}\left[\sigma_j^\alpha\right]\\
&\leqslant c\, 2^{(1+\gamma)\alpha} \sum_{j=0}^\infty 2^{j\gamma\alpha} 3^n\, 2^{-j\beta}\\
&= c\, 2^{(1+\gamma)\alpha} 3^n \sum_{j=0}^\infty 2^{j(\gamma\alpha-\beta)} < \infty.
\end{aligned}$$

The rest of the proof continues literally as on page 154, line 10 onwards.

<u>Alternative Solution:</u> use the subadditivity of $Z \mapsto \mathbb{E}(|Z|^\alpha)$ directly in the second part of the calculation, replacing $\|Z\|_{L^\alpha}$ by $\mathbb{E}(|Z|^\alpha)$.

Problem 10.4 (Solution) We show the following

Theorem. *Let $(B_t)_{t\geqslant 0}$ be a BM[1]. Then $t \mapsto B_t(\omega)$ is for almost all $\omega \in \Omega$ nowhere Hölder continuous of any order $\alpha > 1/2$.*

Proof. Set for every $n \geqslant 1$

$$A_n := A_{n,\alpha} = \left\{\omega \in \Omega : B(\cdot,\omega) \text{ is in } [0,n] \text{ nowhere Hölder continuous of order } \alpha > \tfrac12\right\}.$$

It is not clear if the set $A_{n,\alpha}$ is measurable. We will show that $\Omega \setminus A_{n,\alpha} \subset N_{n,\alpha}$ for a measurable null set $N_{n,\alpha}$.

Assume that the function f is α-Hölder continuous of order α at the point $t_0 \in [0,n]$. Then

$$\exists\, \delta > 0\ \exists\, L > 0\ \forall\, t \in \mathbb{B}(t_0,\delta) : |f(t)-f(t_0)| \leqslant L\,|t-t_0|^\alpha.$$

Since $[0,n]$ is compact, we can use a covering argument to get a uniform Hölder constant. Consider for sufficiently large values of $k \geqslant 1$ the grid $\{\frac{j}{k} : j = 1,\dots,nk\}$. Then there exists a smallest index $j = j(k)$ such that for $\nu \geqslant 3$ and, actually, $1-\nu\alpha+\nu/2 < 0$

$$t_0 \leqslant \frac{j}{k} \quad\text{and}\quad \frac{j}{k},\dots,\frac{j+\nu}{k} \in \mathbb{B}(t_0,\delta).$$

For $i = j+1, j+2, \dots, j+\nu$ we get therefore

$$\begin{aligned}
\left|f\left(\tfrac{i}{k}\right) - f\left(\tfrac{i-1}{k}\right)\right| &\leqslant \left|f\left(\tfrac{i}{k}\right) - f(t_0)\right| + \left|f(t_0) - f\left(\tfrac{i-1}{k}\right)\right| \\
&\leqslant L\left(\left|\tfrac{i}{k} - t_0\right|^\alpha + \left|\tfrac{i-1}{k} - t_0\right|^\alpha\right) \\
&\leqslant L\left(\tfrac{(\nu+1)^\alpha}{k^\alpha} + \tfrac{\nu^\alpha}{k^\alpha}\right) = \tfrac{2L(\nu+1)^\alpha}{k^\alpha}.
\end{aligned}$$

If f is a Brownian path, this implies that for the sets

$$C_m^{L,\nu,\alpha} := \bigcap_{k=m}^{\infty} \bigcup_{j=1}^{kn} \bigcap_{i=j+1}^{j+\nu} \left\{\left|B\left(\tfrac{i}{k}\right) - B\left(\tfrac{i-1}{k}\right)\right| \leqslant \tfrac{2L(\nu+1)^\alpha}{k^\alpha}\right\}$$

we have

$$\Omega \setminus A_{n,\alpha} \subset \bigcup_{L=1}^{\infty} \bigcup_{m=1}^{\infty} C_m^{L,\nu,\alpha}.$$

Our assertion follows if we can show that $\mathbb{P}(C_m^{L,\nu,\alpha}) = 0$ for all $m, L \geqslant 1$ and all rational $\alpha > 1/2$. If $k \geqslant m$,

$$\begin{aligned}
\mathbb{P}(C_m^{L,\nu,\alpha}) &\leqslant \mathbb{P}\left(\bigcup_{j=1}^{kn} \bigcap_{i=j+1}^{j+\nu} \left\{\left|B\left(\tfrac{i}{k}\right) - B\left(\tfrac{i-1}{k}\right)\right| \leqslant \tfrac{2L(\nu+1)^\alpha}{k^\alpha}\right\}\right) \\
&\leqslant \sum_{j=1}^{kn} \mathbb{P}\left(\bigcap_{i=j+1}^{j+\nu} \left\{\left|B\left(\tfrac{i}{k}\right) - B\left(\tfrac{i-1}{k}\right)\right| \leqslant \tfrac{2L(\nu+1)^\alpha}{k^\alpha}\right\}\right) \\
&\overset{\text{(B1)}}{=} \sum_{j=1}^{kn} \mathbb{P}\left(\left\{\left|B\left(\tfrac{i}{k}\right) - B\left(\tfrac{i-1}{k}\right)\right| \leqslant \tfrac{2L(\nu+1)^\alpha}{k^\alpha}\right\}\right)^\nu \\
&\overset{\text{(B2)}}{=} kn\, \mathbb{P}\left(\left\{\left|B\left(\tfrac{1}{k}\right)\right| \leqslant \tfrac{2L(\nu+1)^\alpha}{k^\alpha}\right\}\right)^\nu \\
&\leqslant kn\left(\frac{c}{k^{\alpha-1/2}}\right)^\nu = c^\nu\, n\, k^{1-\nu\alpha+\nu/2} \xrightarrow[k\to\infty]{1-\nu\alpha+\nu/2<0} 0.
\end{aligned}$$

For the last estimate we use $B(\frac{1}{k}) \sim k^{-1/2}B(1)$, cf. 2.12, and therefore

$$\mathbb{P}\left(\left|B\left(\tfrac{1}{k}\right)\right| \leqslant x\right) = \mathbb{P}\left(|B(1)| \leqslant x\sqrt{k}\right) = \frac{1}{\sqrt{2\pi}} \int_{-x\sqrt{k}}^{x\sqrt{k}} \underbrace{e^{-y^2/2}}_{\leqslant 1}\, dy \leqslant c\,x\sqrt{k}.$$

This proves that a Brownian path is almost surely nowhere not Hölder continuous of a fixed order $\alpha > 1/2$. Call the set where this holds Ω_α. Then $\Omega_0 := \bigcap_{\mathbb{Q}\ni\alpha>1/2} \Omega_\alpha$ is a set with $\mathbb{P}(\Omega_0) = 1$ and for all $\omega \in \Omega_0$ we know that BM is nowhere Hölder continuous of any order $\alpha > 1/2$.

The last conclusion uses the following simple remark. Let $0 < \alpha < q < \infty$. Then we have for $f : [0, n] \to \mathbb{R}$ and $x, y \in [0, n]$ with $|x - y| < 1$ that

$$|f(x) - f(y)| \leqslant L|x - y|^q \leqslant L|x - y|^\alpha.$$

Thus q-Hölder continuity implies α-Hölder continuity. ∎

Problem 10.5 (Solution) Fix $\epsilon > 0$, fix a set $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ and $h_0 = h_0(2, \omega)$ such that (10.6) holds for all $\omega \in \Omega_0$, i.e. for all $h \leqslant h_0$ we have

$$\sup_{0 \leqslant t \leqslant 1-h} |B(t+h, \omega) - B(t, \omega)| \leqslant 2\sqrt{2h \log \tfrac{1}{h}}.$$

Pick a partition $\Pi = \{t_0 = 0 < t_1 < \ldots < t_n\}$ of $[0,1]$ with mesh size $h = \max_j (t_j - t_{j-1}) \leqslant h_0$ and assume that $h_0/2 \leqslant h \leqslant h_0$. Then we get

$$\begin{aligned}\sum_{j=1}^n |B(t_j, \omega) - B(t_{j-1}, \omega)|^{2+2\epsilon} &\leqslant 2^{2+2\epsilon} \cdot 2^{1+\epsilon} \sum_{j=1}^n \Big((t_j - t_{j-1}) \log \tfrac{1}{t_j - t_{j-1}} \Big)^{1+\epsilon} \\ &\leqslant c_\epsilon \sum_{j=1}^n (t_j - t_{j-1}) = c_\epsilon.\end{aligned}$$

This shows that

$$\sup_{|\Pi| \leqslant h_0} \sum_{j=1}^n |B(t_j, \omega) - B(t_{j-1}, \omega)|^{2+2\epsilon} \leqslant c_\epsilon.$$

Since we have $|x-y|^p \leqslant 2^{p-1}(|x-z|^p + |z-y|^p)$ and since we can refine any partition Π of $[0,1]$ in finitely many steps to a partition of mesh $< h_0$, we get

$$\mathrm{VAR}_{2+2\epsilon}(B; 1) = \sup_{\Pi \subset [0,1]} \sum_{j=1}^n |B(t_j, \omega) - B(t_{j-1}, \omega)|^{2+2\epsilon} < \infty$$

for all $\omega \in \Omega_0$.

11 The Growth of Brownian Paths

11.1. Fix $C > 2$ and define $A_n := \left\{M_n > C\sqrt{n \log n}\right\}$. By the reflection principle we find

$$\begin{aligned}
\mathbb{P}(A_n) &= \mathbb{P}\left(\sup_{s \leqslant n} B_s > C\sqrt{n \log n}\right) \\
&= 2\,\mathbb{P}\left(B_n > C\sqrt{n \log n}\right) \\
&\overset{\text{scaling}}{=} 2\,\mathbb{P}\left(\sqrt{n}\,B_1 > C\sqrt{n \log n}\right) \\
&= 2\,\mathbb{P}\left(B_1 > C\sqrt{\log n}\right) \\
&\overset{(11.1)}{\leqslant} \frac{2}{\sqrt{2\pi}}\,\frac{1}{C\sqrt{\log n}}\,\exp\left(-\tfrac{C^2}{2}\,\log n\right) \\
&= \frac{2}{\sqrt{2\pi}}\,\frac{1}{C\sqrt{\log n}}\,\frac{1}{n^{C^2/2}}.
\end{aligned}$$

Since $C^2/2 > 2$, the series $\sum_n \mathbb{P}(A_n)$ converges and, by the Borel–Cantelli lemma we see that

$$\exists \Omega_C \subset \Omega,\ \mathbb{P}(\Omega_C) = 1, \quad \forall \omega \in \Omega_C \quad \exists n_0(\omega) \quad \forall n \geqslant n_0(\omega) : M_n(\omega) \leqslant C\sqrt{n \log n}.$$

This shows that

$$\forall \omega \in \Omega_C : \varlimsup_{n\to\infty} \frac{M_n}{\sqrt{n \log n}} \leqslant C.$$

Since every t is in some interval $[n-1, n]$ and since $t \mapsto \sqrt{t \log t}$ is increasing, we see that

$$\frac{M_t}{\sqrt{t \log t}} \leqslant \frac{M_n}{\sqrt{(n-1)\log(n-1)}} = \frac{M_n}{\sqrt{n \log n}}\,\underbrace{\frac{\sqrt{n \log n}}{\sqrt{(n-1)\log(n-1)}}}_{\to 1 \text{ as } n\to\infty}$$

and the claim follows.

<u>*Remark:*</u> We can get the exceptional set in a uniform way: On the set $\Omega_0 := \bigcap_{\mathbb{Q}\ni C>2} \Omega_C$ we have $\mathbb{P}(\Omega_0) = 1$ and

$$\forall \omega \in \Omega_0 : \varlimsup_{n\to\infty} \frac{M_n}{\sqrt{n \log n}} \leqslant 2.$$

11.2. One should assume that $\xi > 0$. Since $y \mapsto \exp(\xi y)$ is monotone increasing, we see

$$\begin{aligned}
\mathbb{P}\left(\sup_{s\leqslant t}(B_s - \tfrac{1}{2}\xi s) > x\right) &= \mathbb{P}\left(e^{\sup_{s\leqslant t}(\xi B_s - \frac{1}{2}\xi^2 s)} > e^{\xi x}\right) \\
&\underset{(A.13)}{\overset{\text{Doob}}{\leqslant}} e^{-x\xi}\,\mathbb{E}\,e^{\xi B_t - \frac{1}{2}\xi^2 t} = e^{-x\xi}.
\end{aligned}$$

(Remark: we have shown (A.13) only for $\sup_{D\ni s\leqslant t} M_s^\xi$ where D is a dense subset of $[0,\infty)$. Since $s\mapsto M_s^\xi$ has continuous paths, it is easy to see that $\sup_{D\ni s\leqslant t} M_s^\xi = \sup_{s\leqslant t} M_s^\xi$ almost surely.)

Usage in step 1° of the Proof of Theorem 11.1: With the notation of the proof we set

$$t = q^n \quad\text{and}\quad \xi = q^{-n}(1+\epsilon)\sqrt{2q^n\log\log q^n} \quad\text{and}\quad x = \frac{1}{2}\sqrt{2q^n\log\log q^n}.$$

Since $\sup_{s\leqslant t}(B_s - \frac12\xi s) \geqslant \sup_{s\leqslant t} B_s - \frac12\xi t$ the above inequality becomes

$$\mathbb{P}\left(\sup_{s\leqslant t} B_s > x + \tfrac12\xi t\right) \leqslant e^{-x\xi}$$

and if we plug in t,x,ξ we see

$$\begin{aligned}
\mathbb{P}\left(\sup_{s\leqslant t} B_s > x + \tfrac12\xi t\right) &= \mathbb{P}\left(\sup_{s\leqslant q^n} B_s > \tfrac12\sqrt{2q^n\log\log q^n} + \tfrac12(1+\epsilon)\sqrt{2q^n\log\log q^n}\right)\\
&= \mathbb{P}\left(\sup_{s\leqslant q^n} B_s > (1+\tfrac{\epsilon}{2})\sqrt{2q^n\log\log q^n}\right)\\
&\leqslant \exp\left(-\tfrac12\sqrt{2q^n\log\log q^n}\, q^{-n}(1+\epsilon)\sqrt{2q^n\log\log q^n}\right)\\
&= \exp\left(-(1+\epsilon)\,\log\log q^n\right)\\
&= \frac{1}{(\log q^n)^{1+\epsilon}}\\
&= \frac{1}{(\log q)^{1+\epsilon}}\,\frac{1}{n^{1+\epsilon}}.
\end{aligned}$$

Now we can argue as in the proof of Theorem 11.1.

11.3. Actually, the hint is not needed, the present proof can be adapted in an easier way. We perform the following changes at the beginning of page 166: *Since every $t>1$ is in some interval of the form $[q^{n-1},q^n]$ and since the function $\Lambda(t)=\sqrt{2t\log\log t}$ is increasing for $t>3$, we find for all $t\geqslant q^{n-1}>3$*

$$\frac{|B(t)|}{\sqrt{2t\log\log t}} \leqslant \frac{\sup_{s\leqslant q^n}|B(s)|}{\sqrt{2q^n\log\log q^n}}\,\frac{\sqrt{2q^n\log\log q^n}}{\sqrt{2q^{n-1}\log\log q^{n-1}}}.$$

Therefore

$$\varlimsup_{t\to\infty}\frac{|B(t)|}{\sqrt{2t\log\log t}} \leqslant (1+\epsilon)\sqrt{q} \quad \textit{a.s.}$$

Letting $\epsilon\to0$ and $q\to1$ along countable sequences, we find the upper bound. □

Remark: The interesting paper by Dupuis [3] shows LILs for processes $(X_t)_{t\geqslant0}$ with stationary and independent increments. It is shown there that the important ingredient are estimates of the type $\mathbb{P}(X_t>x)$. Thus, if we know that $\mathbb{P}(X_t>x)\asymp\mathbb{P}\left(\sup_{s\leqslant t}X_s>x\right)$, we get a LIL for X_t if, and only if, we have a LIL for $\sup_{s\leqslant t}X_s$.

11.4. a) By the LIL for Brownian motion we find

$$\frac{B_t}{b\sqrt{a+t}} = \underbrace{\frac{B_t}{\sqrt{2t\log\log t}}}_{\overline{\lim}_{t\to\infty}(\cdots)=1} \cdot \underbrace{\frac{\sqrt{2t\log\log t}}{b\sqrt{a+t}}}_{\lim_{t\to\infty}(\cdots)=\infty}$$

which shows that

$$\overline{\lim_{t\to\infty}} \frac{B_t}{b\sqrt{a+t}} = \infty$$

almost surely. Therefore, $\mathbb{P}(\tau < \infty) = 1$.

b) Let $b \geqslant 1$ and assume, to the contrary, that $\mathbb{E}\,\tau < \infty$. Then we can use the second Wald identity, cf. Theorem 5.10, and get

$$\mathbb{E}\,\tau = \mathbb{E}\,B^2(\tau) = \mathbb{E}(b^2(a+\tau)) = ab^2 + b^2\,\mathbb{E}\,\tau > b^2\,\mathbb{E}\,\tau \geqslant \mathbb{E}\,\tau,$$

leading to a contradiction. Thus, $\mathbb{E}\,\tau = \infty$.

c) Consider the stopping time $\tau \wedge n$. As in b) we get for all $b > 0$

$$\mathbb{E}\,(\tau\wedge n) = \mathbb{E}\,B^2(\tau\wedge n) \leqslant \mathbb{E}(b^2(a+\tau\wedge n)).$$

This gives, if $b < 1$,

$$(1-b^2)\,\mathbb{E}\,(\tau\wedge n) \leqslant ab^2 \overset{b^2<1}{\Longrightarrow} \mathbb{E}\,(\tau\wedge n) \leqslant \frac{ab^2}{1-b^2} \underset{\text{convergence}}{\overset{\text{monotone}}{\Longrightarrow}} \mathbb{E}\,\tau \leqslant \frac{ab^2}{1-b^2} < \infty.$$

12 Strassen's Functional Law of the Iterated Logarithm

Problem 12.1 (Solution) We construct a counterexample.

The function $w(t) = \sqrt{t}$, $0 \leqslant t \leqslant 1$, is a limit point of the family

$$Z_s(t) = \frac{B(st)}{\sqrt{2s \log\log s}}$$

where $t > 0$ is fixed and for $s \to \infty$.

By the Khintchine's LIL (cf. Theorem 11.1) we obtain

$$\overline{\lim_{s\to\infty}} \frac{B(st)}{\sqrt{2st \log\log(st)}} = 1 \quad \text{(almost surely } \mathbb{P}\text{)}$$

and so

$$\overline{\lim_{s\to\infty}} \frac{B(st)}{\sqrt{2s \log\log(st)}} = \sqrt{t} \quad \text{(almost surely } \mathbb{P}\text{)}$$

which implies

$$\overline{\lim_{s\to\infty}} \frac{B(st)}{\sqrt{2s \log\log s}} = \overline{\lim_{s\to\infty}} \frac{B(st)}{\sqrt{2s \log\log(st)}} \cdot \underbrace{\sqrt{\frac{\log\log(st)}{\log\log s}}}_{\to 1 \text{ for } s \to \infty} = \sqrt{t}.$$

On the other hand, the function $w(t) = \sqrt{t}$ cannot be a limit point of $Z_s(\cdot)$ in $\mathcal{C}_{(o)}[0,1]$ for $s \to \infty$. We prove this indirectly: Let $s_n = s_n(\omega)$ be a sequence, such that $\lim_{n\to\infty} s_n = \infty$. Then

$$\|Z_{s_n}(\cdot) - w(\cdot)\|_\infty \xrightarrow[n\to\infty]{} 0$$

implies that for every $\epsilon > 0$ the inequality

$$(\sqrt{t} - \epsilon) \cdot \sqrt{2s_n \log\log s_n} \leqslant B(s_n \cdot t) \leqslant (\sqrt{t} + \epsilon)\sqrt{2s_n \log\log s_n} \tag{*}$$

holds for all sufficiently large n and *every* $t \in [0,1]$. This, however, contradicts

$$(1-\epsilon)\sqrt{2t_k \log\left(\log \frac{1}{t_k}\right)} \leqslant B(t_k) \leqslant (1+\epsilon)\sqrt{2t_k \log\left(\log \frac{1}{t_k}\right)}, \tag{**}$$

for a sequence $t_k = t_k(\omega) \to 0$, $k \to \infty$, cf. Corollary 11.2.

Indeed: fix some n, then the right side of (*) is in contradiction with the left side of (**).

Remark: Note that

$$\int_0^t w'(s)^2\,ds = \frac{1}{4}\int_0^1 \frac{ds}{s} = +\infty.$$

Problem 12.2 (Solution) For any $w \in \mathcal{K}$ we have

$$|w(t)|^2 = \left|\int_0^t w'(s)\,ds\right|^2 \leqslant \int_0^t w'(s)^2\,ds \cdot \int_0^t 1\,ds \leqslant \int_0^1 w'(s)^2\,ds \cdot t \leqslant t.$$

Problem 12.3 (Solution) Since u is absolutely continuous (w.r.t. Lebesgue measure), for almost all $t \in [0,1]$, the derivative $u'(t)$ exists almost everywhere.

Let t be a point where u' exists and let $(\Pi_n)_{n\geqslant 1}$ be a sequence of partitions of $[0,1]$ such that $|\Pi_n| \to 0$ as $n \to \infty$. We denote the points in Π_n by $t_k^{(n)}$. Clearly, there exists a sequence $(t_{j_n}^{(n)})_{n\geqslant 1}$ such that $t_{j_n}^{(n)} \in \Pi_n$ and $t_{j_n-1}^{(n)} \leqslant t \leqslant t_{j_n}^{(n)}$ for all $n \in \mathbb{N}$ and $t_{j_n}^{(n)} - t_{j_n-1}^{(n)} \to 0$ as $n \to \infty$. We obtain

$$f_n(t) = \left[\frac{1}{t_{j_n}^{(n)} - t_{j_n-1}^{(n)}} \int_{t_{j_n-1}^{(n)}}^{t_{j_n}^{(n)}} u'(s)\,ds\right]^2$$

to simplify notation, we set $t_j := t_{j_n}^{(n)}$ and $t_{j-1} := t_{j_n-1}^{(n)}$, then

$$\begin{aligned}
&= \left[\frac{1}{t_j - t_{j-1}} \cdot \big(u(t_j) - u(t_{j-1})\big)\right]^2 \\
&= \left[\frac{1}{t_j - t_{j-1}} \cdot \big(u(t_j) - u(t) + u(t) - u(t_{j-1})\big)\right]^2 \\
&= \Bigg[\frac{t_j - t}{t_j - t_{j-1}} \cdot \underbrace{\frac{u(t_j) - u(t)}{t_j - t}}_{\to u'(t)} + \frac{t - t_{j-1}}{t_j - t_{j-1}} \cdot \underbrace{\frac{u(t) - u(t_{j-1})}{t - t_{j-1}}}_{\to u'(t)}\Bigg]^2 \\
&\xrightarrow[n\to\infty]{} \big[u'(t)\big]^2.
\end{aligned}$$

Problem 12.4 (Solution) We use the notation of Chapter 4: $\Omega = \mathcal{C}_{(\mathrm{o})}[0,1]$, $w = \omega$, $\mathcal{A} = \mathcal{B}(\mathcal{C}_{(\mathrm{o})}[0,1])$, $\mathbb{P} = \mu$, $B(t,\omega) = B_t(\omega) = w(t)$, $t \in [0,\infty)$.

Linearity of G^ϕ is clear. Let Π_n, $n \geqslant 1$, be a sequence of partitions of $[0,1]$ such that $\lim_{n\to\infty} |\Pi_n| = 0$,

$$\Pi_n = \left\{s_k^{(n)} \,:\, 0 = s_0^{(n)} < s_1^{(n)} < \ldots < s_{l_n}^{(n)} = 1\right\};$$

by $\tilde{s}_k^{(n)}$, $k = 1,\ldots,l_n$ we denote arbitrary intermediate points, i.e. $s_{k-1}^{(n)} \leqslant \tilde{s}_k^{(n)} \leqslant s_k^{(n)}$ for all k. Then we have

$$\begin{aligned}
G^\phi(\omega) &= \phi(1)B_1(\omega) - \int_0^1 B_s(\omega)\,d\phi(s) \\
&= \phi(1)B_1(\omega) - \lim_{|\Pi_n|\to 0} \sum_{k=1}^{l_n} B_{\tilde{s}_k^{(n)}}(\omega)\big(\phi(s_k^{(n)}) - \phi(s_{k-1}^{(n)})\big).
\end{aligned}$$

Write

$$\begin{aligned}
G_n^\phi &:= \phi(1)B_1 - \sum_{k=1}^{l_n} B_{\tilde{s}_k^{(n)}}\big(\phi(s_k^{(n)}) - \phi(s_{k-1}^{(n)})\big) \\
&= \sum_{k=1}^{l_n} (B_1 - B_{\tilde{s}_k^{(n)}})\big(\phi(s_k^{(n)}) - \phi(s_{k-1}^{(n)})\big) + B_1\phi(0).
\end{aligned}$$

Then $G^\phi(\omega) = \lim_{n\to\infty} G_n^\phi(\omega)$ for all $\omega \in \Omega$. Moreover, the elementary identity

$$\sum_{k=1}^{l} a_k(b_k - b_{k-1}) = \sum_{k=1}^{l-1}(a_k - a_{k+1})b_k + a_l b_l - a_1 b_0$$

implies

$$\begin{aligned}
G_n^\phi &= \sum_{k=1}^{l_n-1}(B_{\tilde{s}_{k+1}^{(n)}} - B_{\tilde{s}_k^{(n)}})\phi(s_k^{(n)}) + (B_1 - B_{\tilde{s}_{l_n}^{(n)}})\phi(1) - (B_1 - B_{\tilde{s}_1^{(n)}})\phi(0) + B_1\phi(0) \\
&= \sum_{k=0}^{l_n}(B_{\tilde{s}_{k+1}^{(n)}} - B_{\tilde{s}_k^{(n)}})\phi(s_k^{(n)}) + B_{\tilde{s}_1^{(n)}}\phi(0),
\end{aligned}$$

where $\tilde{s}_{l_n+1}^{(n)} := 1$, $\tilde{s}_0^{(n)} := 0$.

a) G_n^ϕ is a Gaussian random variable with mean $\mathbb{E}\, G_n^\phi = 0$ and variance

$$\begin{aligned}
\mathbb{V}\, G_n^\phi &= \sum_{k=0}^{l_n}\phi^2(s_k^{(n)})\,\mathbb{V}(B_{\tilde{s}_{k+1}^{(n)}} - B_{\tilde{s}_k^{(n)}}) + \phi^2(0)\,\mathbb{V}\, B_{\tilde{s}_1^{(n)}} \\
&= \sum_{k=0}^{l_n}\phi^2(s_k^{(n)})(\tilde{s}_{k+1}^{(n)} - \tilde{s}_k^{(n)}) + \phi^2(0)\tilde{s}_1^{(n)} \\
&\xrightarrow[n\to\infty]{} \int_0^1 \phi^2(s)\,ds.
\end{aligned}$$

This and $\lim_{n\to\infty} G_n^\phi = G^\phi$ ($\mathbb{P}$-a.s.) imply that G^ϕ is a Gaussian random variable with $\mathbb{E}\, G^\phi = 0$ and $\mathbb{V}\, G^\phi = \int_0^1 \phi^2(s)\,ds$.

b) Without loss of generality we use for ϕ and ψ the same sequence of partitions.

Clearly, $G_n^\phi \cdot G_n^\psi \to G^\phi \cdot G^\psi$ for $n \to \infty$ ($\mathbb{P}$-a.s.) Using the elementary inequality $2ab \leqslant a^2 + b^2$ and the fact that for a Gaussian random variable $\mathbb{E}(G^4) = 3(\mathbb{E}(G^2))^2$, we get

$$\begin{aligned}
\mathbb{E}\left((G_n^\phi G_n^\psi)^2\right) &\leqslant \frac{1}{2}\left[\,\mathbb{E}\left((G_n^\phi)^4\right) + \mathbb{E}\left((G_n^\psi)^4\right)\right] \\
&= \frac{3}{2}\left[\left(\,\mathbb{E}(G_n^\phi)^2\right)^2 + \left(\,\mathbb{E}(G_n^\psi)^2\right)^2\right] \\
&\leqslant \frac{3}{2}\left[\left(\int_0^1 \phi^2(s)\,ds\right)^2 + \left(\int_0^1 \psi^2(s)\,ds\right)^2\right] + \epsilon \quad (n \geqslant n_\epsilon).
\end{aligned}$$

This implies

$$\mathbb{E}(G_n^\phi G_n^\psi) \xrightarrow[n\to\infty]{} \mathbb{E}(G^\phi G^\psi).$$

Moreover,

$$\begin{aligned}
\mathbb{E}(G_n^\phi G_n^\psi) = \mathbb{E}&\left[\left(\sum_{k=0}^{l_n}(B_{\tilde{s}_{k+1}^{(n)}} - B_{\tilde{s}_k^{(n)}})\phi(s_k^{(n)})\right)\cdot\left(\sum_{j=0}^{l_n}(B_{\tilde{s}_{j+1}^{(n)}} - B_{\tilde{s}_j^{(n)}})\psi(s_j^{(n)})\right)\right] \\
&+ \phi(0)\psi(0)\,\mathbb{E}(B_{\tilde{s}_1^{(n)}}^2) + \phi(0)\,\mathbb{E}\left[B_{\tilde{s}_1^{(n)}}\sum_{j=0}^{l_n}(B_{\tilde{s}_{j+1}^{(n)}} - B_{\tilde{s}_j^{(n)}})\psi(s_j^{(n)})\right] \\
&+ \psi(0)\,\mathbb{E}\left[B_{\tilde{s}_1^{(n)}}\sum_{k=0}^{l_n}(B_{\tilde{s}_{k+1}^{(n)}} - B_{\tilde{s}_k^{(n)}})\psi(s_k^{(n)})\right]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{l_n} \underbrace{\mathbb{E}\left((B_{\tilde{s}_{k+1}^{(n)}} - B_{\tilde{s}_k^{(n)}})^2\right)}_{=\tilde{s}_{k+1}^{(n)} - \tilde{s}_k^{(n)}} \phi(s_k^{(n)})\psi(s_k^{(n)}) + \cdots \\
&\xrightarrow[n\to\infty]{} \int_0^1 \phi(s)\psi(s)\,ds.
\end{aligned}
$$

This proves

$$
\mathbb{E}\left(G^\phi G^\psi\right) = \int_0^1 \phi(s)\psi(s)\,ds.
$$

c) Using a) and b) we see

$$
\begin{aligned}
\mathbb{E}\left[(G_n^\phi - G_n^\psi)^2\right] &= \mathbb{E}\left[(G_n^\phi)^2\right] - 2\,\mathbb{E}\left[G_n^\phi G_n^\psi\right] + \mathbb{E}\left[(G_n^\psi)^2\right] \\
&= \int_0^1 \phi_n^2(s)\,ds - 2\int_0^1 \phi_n(s)\psi_n(s)\,ds + \int_0^1 \psi_n^2(s)\,ds \\
&= \int_0^1 (\phi_n(s) - \psi_n(s))^2\,ds.
\end{aligned}
$$

This and $\phi_n \to \phi$ in L^2 imply that $(G^{\phi_n})_{n\geqslant 1}$ is a Cauchy sequence in $L^2(\Omega, \mathcal{A}, \mathbb{P})$. Consequently, the limit $X = \lim_{n\to\infty} G^{\phi_n}$ exists in L^2. Moreover, as $\phi_n \to \phi$ in L^2, we also obtain that $\int_0^1 \phi_n^2(s)\,ds \to \int_0^1 \phi^2(s)\,ds$.

Since G^{ϕ_n} is a Gaussian random variable with mean 0 and variance $\int_0^1 \phi_n^2(s)\,ds$, we see that G^ϕ is Gaussian with mean 0 and variance $\int_0^1 \phi^2(s)\,ds$.

Finally, we have $\phi_n \to \phi$ and $\psi_n \to \psi$ in $L_2([0,1])$ implying

$$
\mathbb{E}(G^{\phi_n} G^{\psi_n}) \to \mathbb{E}(G^\phi G^\psi)
$$

—see part b)—and

$$
\int_0^1 \phi_n(s)\psi_n(s)\,ds \to \int_0^1 \phi(s)\psi(s)\,ds.
$$

Thus,

$$
\mathbb{E}(G^\phi G^\psi) = \int_0^1 \phi(s)\psi(s)\,ds.
$$

Problem 12.5 (Solution) The vectors (X, Y) in a) – d) are a.s. limits of two-dimensional Gaussian distributions. Therefore, they are also Gaussian. Their mean is clearly 0. The general density of a two-dimensional Gaussian law (with mean zero) is given by

$$
f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_1^2} + \frac{y^2}{\sigma_2^2} - \frac{2\rho\,xy}{\sigma_1\sigma_2}\right)\right\}.
$$

In order to solve the problems we have to determine the variances $\sigma_1^2 = \mathbb{V}\,X$, $\sigma_2^2 = \mathbb{V}\,Y$ and the correlation coefficient $\rho = \dfrac{\mathbb{E}\,XY}{\sigma_1\sigma_2}$. We will use the results of Problem 12.4.

a) $\sigma_1^2 = \mathbb{V}\left(\int_{1/2}^t s^2\,dw(s)\right) = \int_0^1 \mathbb{1}_{[1/2,t]}(s)s^4\,ds = \dfrac{1}{5}\left(t^5 - \dfrac{1}{32}\right)$,

$\sigma_2^2 = \mathbb{V}\,w(1/2) = 1/2$ ($= \mathbb{V}\,B_{1/2}$ cf. canonical model),

$\mathbb{E}\left(\int_{1/2}^t s^2\,dw(s) \cdot w(1/2)\right) = \int_0^1 \mathbb{1}_{[1/2,t]}(s)s^2 \cdot \mathbb{1}_{[0,1/2]}(s)\,ds = 0$

$\implies \rho = 0.$

b) $\sigma_1^2 = \dfrac{1}{5}\left(t^5 - \dfrac{1}{32}\right)$

$\sigma_2^2 = \mathbb{V}\, w(u+1/2) = u + 1/2$

$$\mathbb{E}\left(\int_{1/2}^t s^2\, dw(s) \cdot w(u+1/2)\right)$$
$$= \int_0^1 \mathbb{1}_{[1/2,t]}(s)s^2 \cdot \mathbb{1}_{[0,u+1/2]}(s)\, ds$$
$$= \int_{1/2}^{(1/2+u)\wedge t} s^2\, ds$$
$$= \frac{1}{3}\left(\left(\left(\frac{1}{2}+u\right)\wedge t\right)^3 - \frac{1}{8}\right).$$
$$\implies \rho = \frac{\frac{1}{3}\left(\left(\left(\frac{1}{2}+u\right)\wedge t\right)^3 - \frac{1}{8}\right)}{\left[\frac{1}{5}\left(t^5-\frac{1}{32}\right)\cdot\left(u+\frac{1}{2}\right)\right]^{1/2}}.$$

c) $\sigma_1^2 = \mathbb{V}\left(\displaystyle\int_{1/2}^t s^2\, dw(s)\right) = \dfrac{1}{5}\left(t^5 - \dfrac{1}{32}\right),$

$$\sigma_2^2 = \mathbb{V}\left(\int_{1/2}^t s\, dw(s)\right) = \frac{1}{3}\left(t^3 - \frac{1}{8}\right)$$
$$\mathbb{E}\left(\int_{1/2}^t s^2\, dw(s)\cdot \int_{1/2}^t s\, dw(s)\right) = \int_{1/2}^t s^3\, ds = \frac{1}{4}\left(t^4 - \frac{1}{16}\right)$$
$$\implies \rho = \frac{\frac{1}{4}\left(t^4-\frac{1}{16}\right)}{\left[\frac{1}{5}\left(t^5-\frac{1}{32}\right)\cdot\frac{1}{3}\left(t^3-\frac{1}{8}\right)\right]^{1/2}}.$$

d) $\sigma_1^2 = \mathbb{V}\left(\displaystyle\int_{1/2}^1 e^s\, dw(s)\right) = \displaystyle\int_{1/2}^1 e^{2s}\, ds = \dfrac{1}{2}(e^2 - e),$

$$\sigma_2^2 = \mathbb{V}(w(1) - w(1/2)) = 1/2,$$
$$\mathbb{E}\left(\int_{1/2}^1 e^s\, dw(s)\cdot (w(1)-w(1/2))\right) = \int_{1/2}^1 e^s\cdot 1\, ds = e - e^{1/2}.$$
$$\implies \rho = \frac{e - e^{1/2}}{\left(\frac{1}{4}\left(e^2-e\right)\right)^{1/2}}.$$

Problem 12.6 (Solution) Let $w_n \in F$, $n \geqslant 1$, and $w_n \to v$ in $\mathcal{C}_{(o)}[0,1]$. We have to show that $v \in F$.

Now:

$$w_n \in F \implies \exists (c_n, r_n) \in [q^{-1},1]\times[0,1] \,:\, |w_n(c_n r_n) - w_n(r_n)| \geqslant 1.$$

Observe that the function $(c,r)\mapsto w(cr) - w(r)$ with $(c,r)\in[q^{-1},1]\times[0,1]$ is continuous for every $w \in \mathcal{C}_{(o)}[0,1]$.

Since $[q^{-1},1]\times[0,1]$ is compact, there exists a subsequence $(n_k)_{k\geqslant 1}$ such that $c_{n_k}\to\tilde{c}$ and $r_{n_k}\to\tilde{r}$ as $k\to\infty$ and $(\tilde{c},\tilde{r})\in[q^{-1},1]\times[0,1]$.

By assumption, $w_{n_k}\to v$ uniformly and this implies

$$w_{n_k}(c_{n_k} r_{n_k}) \to v(\tilde{c}\tilde{r}) \quad\text{and}\quad w_{n_k}(r_{n_k}) \to v(\tilde{r}).$$

Finally,

$$|v(\tilde{c}\tilde{r}) - v(\tilde{r})| = \lim_{k\to\infty} |w_{n_k}(c_{n_k} r_{n_k}) - w_{n_k}(r_{n_k})| \geqslant 1,$$

and $v \in F$ follows.

Problem 12.7 (Solution) Set $L(t) = \sqrt{2t \log\log t}$, $t \geqslant e$ and $s_n = q^n$, $n \in \mathbb{N}$, $q > 1$. Then:

a) for the first inequality:

$$\begin{aligned}
\mathbb{P}\left(\frac{|B(s_{n-1})|}{L(s_n)} > \frac{\epsilon}{4}\right) &= \mathbb{P}\left(\underbrace{\left|\frac{B(s_{n-1})}{\sqrt{s_{n-1}}}\right|}_{\sim N(0,1)} \cdot \frac{1}{\sqrt{2q \log\log s_n}} > \frac{\epsilon}{4}\right)\\
&= \mathbb{P}\left(|B(1)| > \frac{\epsilon}{4}\sqrt{2q\log\log q^n}\right)
\end{aligned}$$

using Problem 9.6 and $\mathbb{P}(|Z| > x) = 2\,\mathbb{P}(Z > x)$ for $x \geqslant 0$

$$\begin{aligned}
&\leqslant \sqrt{\frac{2}{\pi}}\,\frac{4}{\epsilon\sqrt{2q\log\log q^n}} \cdot \exp\left\{-\frac{\epsilon^2}{32}\cdot 2q\log\log q^n\right\}\\
&\leqslant \frac{C}{n^2}
\end{aligned}$$

if q is sufficiently large.

b) for the second inequality:

$$\begin{aligned}
\sup_{t\leqslant q^{-1}} |w(t)| &= \sup_{t\leqslant q^{-1}} \left|\int_0^t w'(s)\,ds\right|\\
&\leqslant \int_0^{1/q} |w'(s)|\,ds\\
&\leqslant \left[\int_0^{1/q} w'(s)^2\,ds \cdot \int_0^{1/q} ds\right]^{1/2} \cdot \left[\int_0^1 w'(s)^2\,ds \cdot \frac{1}{q}\right]^{1/2}\\
&\leqslant \sqrt{\frac{r}{q}} < \frac{\epsilon}{4}
\end{aligned}$$

for all sufficiently large q.

c) for the third inequality: Brownian scaling $\frac{B(\cdot\, s_n)}{\sqrt{s_n}} \sim B(\cdot)$ yields

$$\begin{aligned}
\mathbb{P}\left(\sup_{0\leqslant t\leqslant q^{-1}} \frac{|B(ts_n)|}{\sqrt{2s_n\log\log s_n}} > \frac{\epsilon}{4}\right) &= \mathbb{P}\left(\sup_{0\leqslant t\leqslant q^{-1}} \frac{|B(t)|}{\sqrt{2\log\log s_n}} > \frac{\epsilon}{4}\right)\\
&= \mathbb{P}\left(\sup_{0\leqslant t\leqslant q^{-1}} |B(t)| > \frac{\epsilon}{4}\sqrt{2\log\log s_n}\right)\\
&\leqslant 2\,\mathbb{P}\left(|B(1/q)| > \frac{\epsilon}{4}\sqrt{2\log\log s_n}\right)\\
&\overset{(*)}{\leqslant} 2\,\mathbb{P}\left(\frac{|B(1/q)|}{\sqrt{1/q}} > \frac{\epsilon}{4}\sqrt{2q\log\log q^n}\right) \leqslant \frac{C}{n^2}
\end{aligned}$$

for all q sufficiently large. In the estimate marked with (*) we used

$$\mathbb{P}\big(\sup_{0\leqslant t\leqslant t_0} |B(t)| > x\big) \leqslant 2\,\mathbb{P}\big(\sup_{0\leqslant t\leqslant t_0} B(t) > x\big) \overset{\text{Thm.}}{\underset{6.9}{=}} 2\,\mathbb{P}(M(t_0) > x) = 2\,\mathbb{P}\big(|B(t_0)| > x\big).$$

d) for the last inequality:

$$
\begin{aligned}
&\mathbb{P}\left(\frac{|B(s_{n-1})|}{L(s_n)} + \sup_{t\leqslant q^{-1}}|w(t)| + \sup_{0\leqslant t\leqslant q^{-1}}\frac{|B(ts_n)|}{L(s_n)} > \frac{3\epsilon}{4}\right)\\
&\quad\leqslant \mathbb{P}\left(\frac{|B(s_{n-1})|}{L(s_n)} > \frac{\epsilon}{4} \quad\text{or}\quad \sup_{t\leqslant q^{-1}}|w(t)| > \frac{\epsilon}{4} \quad\text{or}\quad \sup_{0\leqslant t\leqslant q^{-1}}\frac{|B(ts_n)|}{L(s_n)} > \frac{\epsilon}{4}\right)\\
&\quad\leqslant \mathbb{P}\left(\frac{|B(s_{n-1})|}{L(s_n)} > \frac{\epsilon}{4}\right) + \mathbb{P}\left(\sup_{t\leqslant q^{-1}}|w(t)| > \frac{\epsilon}{4}\right) + \mathbb{P}\left(\sup_{0\leqslant t\leqslant q^{-1}}\frac{|B(ts_n)|}{L(s_n)} > \frac{\epsilon}{4}\right)\\
&\quad\leqslant \frac{C}{n^2} + 0 + \frac{C}{n^2}
\end{aligned}
$$

for all sufficiently large q. Using the Borel–Cantelli lemma we see that

$$
\overline{\lim_{n\to\infty}}\left(\frac{|B(s_{n-1})|}{L(s_n)} + \sup_{t\leqslant q^{-1}}|w(t)| + \sup_{0\leqslant t\leqslant q^{-1}}\frac{|B(ts_n)|}{L(s_n)}\right) \leqslant \frac{3}{4}\epsilon.
$$

13 Skorokhod Representation

Problem 13.1 (Solution) Clearly, $\mathcal{F}_t^B := \sigma(B_r : r \leqslant t) \subset \sigma(B_r : r \leqslant t,\ U, V) = \mathcal{F}_t$. It remains to show that $B_t - B_s \perp\!\!\!\perp \mathcal{F}_s$ for all $s \leqslant t$. Let A, A'', C be Borel sets in $\mathbb{R}^d$. Then we find for $F \in \mathcal{F}_s^B$

$$
\begin{aligned}
&\mathbb{P}\left(\{B_t - B_s \in C\} \cap F \cap \{U \in A\} \cap \{V \in A'\}\right)\\
&\quad= \mathbb{P}\left(\{B_t - B_s \in C\} \cap F\right) \cdot \mathbb{P}\left(\{U \in A\} \cap \{V \in A'\}\right) && (\text{since } U, V \perp\!\!\!\perp \mathcal{F}_\infty^B)\\
&\quad= \mathbb{P}\left(\{B_t - B_s \in C\}\right) \cdot \mathbb{P}\left(F\right) \cdot \mathbb{P}\left(\{U \in A\} \cap \{V \in A'\}\right) && (\text{since } B_t - B_s \perp\!\!\!\perp \mathcal{F}_\infty^B)\\
&\quad= \mathbb{P}\left(\{B_t - B_s \in C\}\right) \cdot \mathbb{P}\left(F \cap \{U \in A\} \cap \{V \in A'\}\right) && (\text{since } U, V \perp\!\!\!\perp \mathcal{F}_\infty^B)
\end{aligned}
$$

and this shows that $B_t - B_s$ is independent of the family $\mathcal{E}_s = \left\{F \cap G : F \in \mathcal{F}_s^B, G \in \sigma(U, V)\right\}$. This family is stable under finite intersections, so $B_t - B_s \perp\!\!\!\perp \sigma(\mathcal{E}_s) = \mathcal{F}_s$.

14 Stochastic Integrals: L^2–Theory

Problem 14.1 (Solution) By definition of the angle bracket,

$$M^2 - \langle M \rangle \quad \text{and} \quad N^2 - \langle N \rangle$$

are martingales. Moreover, $M \pm N$ are L^2-martingales, i.e.

$$(M+N)^2 - \langle M+N \rangle \quad \text{and} \quad (M-N)^2 - \langle M-N \rangle$$

are martingales. So, we subtract them to get a new martingale:

$$(M+N)^2 - (M-N)^2 = 4MN \quad \text{and} \quad \langle M+N \rangle - \langle M-N \rangle \stackrel{\text{def}}{=} 4\langle M, N \rangle$$

which shows that $4MN - 4\langle MN \rangle$ is a martingale.

Problem 14.2 (Solution) Note that

$$[a,b) \cap [c,d) = [a \vee c, b \wedge d) \qquad \text{(with the convention } [M,m) = \varnothing \text{ if } M \geqslant m).$$

Then assume that we have any two representations for a simple process

$$f = \sum_j \phi_{j-1} \mathbb{1}_{[s_{j-1}, s_j)} = \sum_k \psi_{k-1} \mathbb{1}_{[t_{k-1}, t_k)}$$

Then

$$f = \sum_j \phi_{j-1} \mathbb{1}_{[s_{j-1}, s_j)} \mathbb{1}_{[0,T)} = \sum_{j,k} \phi_{j-1} \mathbb{1}_{[s_{j-1}, s_j)} \mathbb{1}_{[t_{k-1}, t_k)}$$

and, similarly,

$$f = \sum_{k,j} \psi_{k-1} \mathbb{1}_{[s_{j-1}, s_j)} \mathbb{1}_{[t_{k-1}, t_k)}.$$

Then we get, since $\phi_{j-1} = \psi_{k-1}$ whenever $[s_{j-1}, s_j) \cap [t_{k-1}, t_k) \neq \varnothing$

$$\begin{aligned}
\sum_j \phi_{j-1}(B(s_j) - B(s_{j-1})) &= \sum\sum_{(j,k):[s_{j-1},s_j)\cap[t_{k-1},t_k)\neq\varnothing} \phi_{j-1}(B(s_j \wedge t_k) - B(s_{j-1} \vee t_{k-1})) \\
&= \sum\sum_{(j,k):[s_{j-1},s_j)\cap[t_{k-1},t_k)\neq\varnothing} \psi_{k-1}(B(s_j \wedge t_k) - B(s_{j-1} \vee t_{k-1})) \\
&= \sum\sum_{(k,j):[s_{j-1},s_j)\cap[t_{k-1},t_k)\neq\varnothing} \psi_{k-1}(B(s_j \wedge t_k) - B(s_{j-1} \vee t_{k-1})) \\
&= \sum_k \psi_{k-1}(B(t_k) - B(t_{k-1}))
\end{aligned}$$

Problem 14.3 (Solution) • Positivity is clear, finiteness follows with Doob's maximal inequality

$$\mathbb{E}\left[\sup_{s\leqslant T}|M_s|^2\right] \leqslant 4 \sup_{s\leqslant T}\mathbb{E}\left[|M_s|^2\right] = 4\,\mathbb{E}\left[|M_T|^2\right].$$

- Triangle inequality:

$$\begin{aligned}\|M+N\|_{\mathcal{M}^2_T} &= \left(\mathbb{E}\left[\sup_{s\leqslant T}|M_s+N_s|^2\right]\right)^{\frac12}\\ &\leqslant \left(\mathbb{E}\left[(\sup_{s\leqslant T}|M_s| + \sup_{s\leqslant T}|N_s|)^2\right]\right)^{\frac12}\\ &\leqslant \left(\mathbb{E}\left[\sup_{s\leqslant T}|M_s|^2\right]\right)^{\frac12} + \left(\mathbb{E}\left[\sup_{s\leqslant T}|N_s|^2\right]\right)^{\frac12}\end{aligned}$$

  where we used in the first estimate the subadditivity of the supremum and in the second inequality the Minkowski inequality (triangle inequality) in L^2.

- Positive homogeneity

$$\|\lambda M\|_{\mathcal{M}^2_T} = \left(\mathbb{E}\left[\sup_{s\leqslant T}|\lambda M_s|^2\right]\right)^{\frac12} = |\lambda|\left(\mathbb{E}\left[\sup_{s\leqslant T}|M_s|^2\right]\right)^{\frac12} = |\lambda|\cdot\|M\|_{\mathcal{M}^2_T}.$$

- Definiteness

$$\|M\|_{\mathcal{M}^2_T} = 0 \iff \sup_{s\leqslant T}|M_s|^2 = 0 \quad \text{(almost surely)}.$$

Problem 14.4 (Solution) Let $f_n \to f$ and $g_n \to f$ be two sequences which approximate f in the norm of $L^2(\lambda_T\otimes\mathbb{P})$. Then we have

$$\begin{aligned}\mathbb{E}\left(\left|f_n\bullet B_T - g_n\bullet B_T\right|^2\right) &= \mathbb{E}\left(\left|(f_n-g_n)\bullet B_T\right|^2\right)\\ &= \mathbb{E}\left(\int_0^T |f_n(s)-g_n(s)|^2\,ds\right)\\ &= \|f_n-g_n\|^2_{L^2(\lambda_T\otimes\mathbb{P})}\\ &\xrightarrow[n\to\infty]{} 0.\end{aligned}$$

This means that

$$L^2(\mathbb{P})\text{-}\lim_{n\to\infty} f_n\bullet B_T = L^2(\mathbb{P})\text{-}\lim_{n\to\infty} g_n\bullet B_T.$$

Problem 14.5 (Solution) Solution 1: Let τ be a stopping time and consider the sequence of discrete stopping times

$$\tau_m := \frac{\lfloor 2^m\tau\rfloor + 1}{2^m}\wedge T.$$

Let $t_0 = 0 < t_1 < t_2 < \ldots < t_n = T$ and, without loss of generality, $\tau_m(\Omega)\subset\{t_0,\ldots,t_n\}$. Then $(B^2_{t_j} - t_j)_j$ is again a discrete martingale and by optional stopping we get that $(B^2_{\tau_m\wedge t_j} - \tau_m\wedge t_j)_j$ is a discrete martingale. This means that for each $m\geqslant 1$

$$\langle B^{\tau_m}\rangle_{t_j} = \tau_m\wedge t_j \quad \text{for all } \ j$$

and this indicates that we can set $\langle B^\tau\rangle_t = t\wedge\tau$. This process makes $B^2_{t\wedge\tau} - t\wedge\tau$ into a martingale. Indeed: fix $0\leqslant s\leqslant t\leqslant T$ and add them to the partition, if necessary. Then

$$B^2_{\tau_m\wedge t} \xrightarrow[m\to\infty]{\text{a.e.}} B_{\tau\wedge t} \quad\text{and}\quad B^2_{\tau_m\wedge t} \xrightarrow[m\to\infty]{L^1(\mathbb{P})} B_{\tau\wedge t}$$

by dominated convergence, since $\sup_{r\leqslant T} B_r^2$ is an integrable majorant. Thus,

$$\int_F B_{\tau\wedge s}\,d\mathbb{P} = \lim_{m\to\infty}\int_F B_{\tau_m\wedge s}\,d\mathbb{P} = \lim_{m\to\infty}\int_F B_{\tau_m\wedge t}\,d\mathbb{P}\int_F B_{\tau\wedge t}\,d\mathbb{P} \quad\text{for all}\quad F\in\mathcal{F}_s$$

and we conclude that $(B^2_{\tau\wedge t} - \tau\wedge t)_t$ is a martingale.

<u>Solution 2:</u> Observe that

$$B_t^\tau = \int_0^t \mathbb{1}_{[0,\tau)}\,dB_s$$

and by Theorem 14.13 b) we get

$$\left\langle \int_0^\bullet \mathbb{1}_{[0,\tau)}\,dB_s\right\rangle_t = \int_0^t \mathbb{1}^2_{[0,\tau)}\,ds = \int_0^t \mathbb{1}_{[0,\tau)}\,ds = \tau\wedge t.$$

(Of course, one should make sure that $\mathbb{1}_{[0,\tau)}\in\mathcal{L}^2_T$, see e.g. Problem 14.14 below or Problem 15.2 in combination with Theorem 14.20.)

Problem 14.6 (Solution) We begin with a general remark: if $f=0$ on $[0,s]\times\Omega$, we can use Theorem 14.13 f) and deduce $f\bullet B_s = 0$.

a) We have

$$\mathbb{E}\left[(f\bullet B_t)^2\,\middle|\,\mathcal{F}_s\right] = \mathbb{E}\left[(f\bullet B_t - f\bullet B_s)^2\,\middle|\,\mathcal{F}_s\right] \overset{14.13\ b)}{\underset{(14.19)}{=}} \mathbb{E}\left[\int_s^t f^2(r)\,dr\,\middle|\,\mathcal{F}_s\right].$$

If both f and g vanish on $[0,s]$, the same is true for $f\pm g$. We get

$$\mathbb{E}\left[\big((f\pm g)\bullet B_t\big)^2\,\middle|\,\mathcal{F}_s\right] = \mathbb{E}\left[\int_s^t (f\pm g)^2(r)\,dr\,\middle|\,\mathcal{F}_s\right].$$

Subtracting the 'minus' version from the 'plus' version and gives

$$\mathbb{E}\left[\big((f+g)\bullet B_t\big)^2 - \big((f-g)\bullet B_t\big)^2\,\middle|\,\mathcal{F}_s\right] = \mathbb{E}\left[\int_s^t (f+g)^2(r) - (f-g)^2(r)\,dr\,\middle|\,\mathcal{F}_s\right].$$

or

$$4\,\mathbb{E}\left[(f\bullet B_t)\cdot(g\bullet B_t)\,\middle|\,\mathcal{F}_s\right] = 4\,\mathbb{E}\left[\int_s^t (f\cdot g)(r)\,dr\,\middle|\,\mathcal{F}_s\right].$$

b) Since $f\bullet B_t$ is a martingale, we get for $t\geqslant s$

$$\mathbb{E}\left(f\bullet B_t\,\middle|\,\mathcal{F}_s\right) \overset{\text{martingale}}{=} f\bullet B_s \overset{\text{see above}}{=} 0$$

since f vanishes on $[0,s]$.

c) By Theorem 14.13 f) we have for all $t\leqslant T$

$$f\bullet B_t(\omega)\mathbb{1}_A(\omega) = 0\bullet B_t(\omega)\mathbb{1}_A(\omega) = 0.$$

Problem 14.7 (Solution) Because of Lemma 14.10 it is enough to show that $f_n \bullet B_T \xrightarrow{n\to\infty} f \bullet B_T$ in $L^2(\mathbb{P})$. This follows immediately from Theorem 14.13 c):

$$\begin{aligned}\mathbb{E}\left[\left|f_n \bullet B_T - f \bullet B_T\right|^2\right] &= \mathbb{E}\left[\left|(f_n - f) \bullet B_T\right|^2\right]\\ &= \mathbb{E}\left[\int_0^T |f_n(s) - f(s)|^2\,ds\right] \xrightarrow{n\to\infty} 0.\end{aligned}$$

Problem 14.8 (Solution) Assume that $(f \bullet B)^2 - A$ is a martingale where A_t is continuous and increasing. Since $(f \bullet B)^2 - f^2 \bullet \langle B\rangle$ is a martingale, we conclude that

$$((f \bullet B)^2 - f^2 \bullet \langle B\rangle) - ((f \bullet B)^2 - A) = f^2 \bullet \langle B\rangle - A$$

is a continuous martingale with BV paths. Hence, it is a.s. constant.

Problem 14.9 (Solution) If $X_n \xrightarrow{L^2} X$ then $\sup_n \mathbb{E}(X_n^2) < \infty$ and the claim follows from the fact that

$$\begin{aligned}\mathbb{E}\,|X_n^2 - X_m^2| &= \mathbb{E}\left[|X_n - X_m||X_n + X_m|\right]\\ &\leqslant \sqrt{\mathbb{E}\,|X_n + X_m|^2}\sqrt{\mathbb{E}\,|X_n - X_m|^2}\\ &\leqslant \left(\sqrt{\mathbb{E}\,|X_n|^2} + \sqrt{\mathbb{E}\,|X_m|^2}\right)\sqrt{\mathbb{E}\,|X_n - X_m|^2}.\end{aligned}$$

Problem 14.10 (Solution) Let $\Pi = \{t_0 = 0 < t_1 < \ldots < t_n = T\}$ be a partition of $[0,T]$. Then we get

$$\begin{aligned}B_T^3 &= \sum_{j=1}^n \left(B_{t_j}^3 - B_{t_{j-1}}^3\right)\\
&= \sum_{j=1}^n \left(B_{t_j} - B_{t_{j-1}}\right)\left[B_{t_j}^2 + B_{t_j}B_{t_{j-1}} + B_{t_{j-1}}^2\right]\\
&= \sum_{j=1}^n \left(B_{t_j} - B_{t_{j-1}}\right)\left[B_{t_j}^2 - 2B_{t_j}B_{t_{j-1}} + B_{t_{j-1}}^2 + 3B_{t_j}B_{t_{j-1}}\right]\\
&= \sum_{j=1}^n \left(B_{t_j} - B_{t_{j-1}}\right)\left[(B_{t_j} - B_{t_{j-1}})^2 + 3B_{t_j}B_{t_{j-1}}\right]\\
&= \sum_{j=1}^n \left(B_{t_j} - B_{t_{j-1}}\right)\left[(B_{t_j} - B_{t_{j-1}})^2 + 3B_{t_{j-1}}^2 + 3B_{t_{j-1}}(B_{t_j} - B_{t_{j-1}})\right]\\
&= \sum_{j=1}^n \left(B_{t_j} - B_{t_{j-1}}\right)^3 + 3\sum_{j=1}^n B_{t_{j-1}}^2\left(B_{t_j} - B_{t_{j-1}}\right) + 3\sum_{j=1}^n B_{t_{j-1}}\left(B_{t_j} - B_{t_{j-1}}\right)^2\\
&= \sum_{j=1}^n \left(B_{t_j} - B_{t_{j-1}}\right)^3 + 3\sum_{j=1}^n B_{t_{j-1}}^2\left(B_{t_j} - B_{t_{j-1}}\right) + 3\sum_{j=1}^n B_{t_{j-1}}(t_j - t_{j-1})\\
&\qquad + 3\sum_{j=1}^n B_{t_{j-1}}\left[\left(B_{t_j} - B_{t_{j-1}}\right)^2 - (t_j - t_{j-1})\right]\\
&= I_1 + I_2 + I_3 + I_4.\end{aligned}$$

Clearly,

$$I_2 \xrightarrow[|\Pi|\to 0]{} 3\int_0^T B_s^2\,dB_s \quad\text{and}\quad I_3 \xrightarrow[|\Pi|\to 0]{} 3\int_0^T B_s\,ds$$

by Proposition 14.16 and by the construction of the stochastic resp. Riemann-Stieltjes integral. The latter also converges in L^2 since I_2 and, as we will see in a moment, I_1 and I_4 converge in L^2-sense.

Let us show that $I_1, I_4 \to 0$.

$$\begin{aligned}
\mathbb{V}\, I_1 = \mathbb{V}\left(\sum_{j=1}^n \left(B_{t_j} - B_{t_{j-1}}\right)^3\right) &\overset{\text{(B1)}}{=} \sum_{j=1}^n \mathbb{V}\left(\left(B_{t_j} - B_{t_{j-1}}\right)^3\right) \\
&\overset{\text{(B2)}}{=} \sum_{j=1}^n \mathbb{V}\left(B^3_{t_j - t_{j-1}}\right) \\
&\overset{\text{scaling}}{=} \sum_{j=1}^n (t_j - t_{j-1})^3\, \mathbb{V}\left(B_1^3\right) \\
&\leqslant |\Pi|^2 \sum_{j=1}^n (t_j - t_{j-1})\, \mathbb{V}\left(B_1^3\right) \\
&= |\Pi|^2 T\, \mathbb{V}(B_1^3) \xrightarrow[|\Pi|\to 0]{} 0.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\mathbb{E}(I_4^2) &= \mathbb{E}\left(\left(3\sum_{j=1}^n B_{t_{j-1}}\left[\left(B_{t_j} - B_{t_{j-1}}\right)^2 - (t_j - t_{j-1})\right]\right)^2\right) \\
&= 9\,\mathbb{E}\left(\sum_{j=1}^n \sum_{k=1}^n B_{t_{j-1}}\left[\left(B_{t_j} - B_{t_{j-1}}\right)^2 - (t_j - t_{j-1})\right] B_{t_{k-1}}\left[\left(B_{t_k} - B_{t_{k-1}}\right)^2 - (t_k - t_{k-1})\right]\right) \\
&= 9\,\mathbb{E}\left(\sum_{j=1}^n B^2_{t_{j-1}}\left[\left(B_{t_j} - B_{t_{j-1}}\right)^2 - (t_j - t_{j-1})\right]^2\right)
\end{aligned}$$

since the mixed terms break away, see below.

$$\begin{aligned}
&= 9\sum_{j=1}^n \mathbb{E}\left(B^2_{t_{j-1}}\left[\left(B_{t_j} - B_{t_{j-1}}\right)^2 - (t_j - t_{j-1})\right]^2\right) \\
&\overset{\text{(B1)}}{=} 9\sum_{j=1}^n \mathbb{E}\left(B^2_{t_{j-1}}\right) \mathbb{E}\left(\left[\left(B_{t_j} - B_{t_{j-1}}\right)^2 - (t_j - t_{j-1})\right]^2\right) \\
&\overset{\text{(B2)}}{=} 9\sum_{j=1}^n \mathbb{E}\left(B^2_{t_{j-1}}\right) \mathbb{E}\left(\left[B^2_{t_j - t_{j-1}} - (t_j - t_{j-1})\right]^2\right) \\
&\overset{\text{scaling}}{=} 9\sum_{j=1}^n t_{j-1}\, \mathbb{E}\left(B_1^2\right)(t_j - t_{j-1})^2\, \mathbb{E}\left(\left[B_1^2 - 1\right]^2\right) \\
&= 9\sum_{j=1}^n t_{j-1}(t_j - t_{j-1})^2\, \mathbb{V}(B_1^2) \\
&\leqslant 9T|\Pi| \sum_{j=1}^n (t_j - t_{j-1})\, \mathbb{V}(B_1^2) \\
&\leqslant 9T^2|\Pi|\, \mathbb{V}(B_1^2) \xrightarrow[|\Pi|\to 0]{} 0.
\end{aligned}$$

Now for the argument with the mixed terms. Let $j < k$; then $t_{j-1} < t_j \leqslant t_{k-1} < t_k$, and by the tower property,

$$\mathbb{E}\left(B_{t_{j-1}}\left[\left(B_{t_j} - B_{t_{j-1}}\right)^2 - (t_j - t_{j-1})\right] B_{t_{k-1}}\left[\left(B_{t_k} - B_{t_{k-1}}\right)^2 - (t_k - t_{k-1})\right]\right)$$

$$
\begin{aligned}
&\overset{\text{tower}}{=} \mathbb{E}\Big(\mathbb{E}\Big[B_{t_{j-1}}\big[\big(B_{t_j}-B_{t_{j-1}}\big)^2-(t_j-t_{j-1})\big]B_{t_{k-1}}\big[\big(B_{t_k}-B_{t_{k-1}}\big)^2-(t_k-t_{k-1})\big]\,\Big|\,\mathcal{F}_{t_{k-1}}\Big]\Big)\\
&\overset{\text{pull out}}{=} \mathbb{E}\Big(B_{t_{j-1}}\big[\big(B_{t_j}-B_{t_{j-1}}\big)^2-(t_j-t_{j-1})\big]B_{t_{k-1}}\ \mathbb{E}\Big[\big[\big(B_{t_k}-B_{t_{k-1}}\big)^2-(t_k-t_{k-1})\big]\,\Big|\,\mathcal{F}_{t_{k-1}}\Big]\Big)\\
&\overset{\text{(B1)}}{=} \mathbb{E}\Big(B_{t_{j-1}}\big[\big(B_{t_j}-B_{t_{j-1}}\big)^2-(t_j-t_{j-1})\big]B_{t_{k-1}}\ \underbrace{\mathbb{E}\big[\big(B_{t_k}-B_{t_{k-1}}\big)^2-(t_k-t_{k-1})\big]}_{=0}\Big)\\
&= 0.
\end{aligned}
$$

Problem 14.11 (Solution) Let $\Pi = \{t_0 = 0 < t_1 < \ldots < t_n = T\}$ be a partition of $[0,T]$. Then we get

$$
\begin{aligned}
&f(t_j)B_{t_j} - f(t_{j-1})B_{t_{j-1}}\\
&\qquad = f(t_{j-1})(B_{t_j}-B_{t_{j-1}}) + B_{t_{j-1}}(f(t_j)-f(t_{j-1})) + (B_{t_j}-B_{t_{j-1}})(f(t_j)-f(t_{j-1})).
\end{aligned}
$$

If we sum over $j = 1,\ldots,n$ we get

$$
\begin{aligned}
&f(T)B_T - f(0)B_0\\
&= \sum_{j=1}^n f(t_{j-1})(B_{t_j}-B_{t_{j-1}}) + \sum_{j=1}^n B_{t_{j-1}}(f(t_j)-f(t_{j-1})) + \sum_{j=1}^n (B_{t_j}-B_{t_{j-1}})(f(t_j)-f(t_{j-1}))\\
&= I_1 + I_2 + I_3.
\end{aligned}
$$

Clearly,

$$
\begin{aligned}
I_1 &\xrightarrow{L^2} \int_0^T f(s)\,dB_s &&\text{(stochastic integral)}\\
I_2 &\xrightarrow{\text{a.s.}} \int_0^T B_s\,df(x) &&\text{(Riemann-Stieltjes integral)}
\end{aligned}
$$

and if we can show that $I_3 \to 0$ in L^2, then we are done (as this also implies the L^2-convergence of I_2). Now we have

$$
\begin{aligned}
&\mathbb{E}\left[\left(\sum_{j=1}^n (B_{t_j}-B_{t_{j-1}})(f(t_j)-f(t_{j-1}))\right)^2\right]\\
&\quad= \mathbb{E}\left[\sum_{j=1}^n\sum_{k=1}^n (B_{t_j}-B_{t_{j-1}})(f(t_j)-f(t_{j-1}))(B_{t_k}-B_{t_{k-1}})(f(t_k)-f(t_{k-1}))\right]
\end{aligned}
$$

the mixed terms break away because of the independent increments property of Brownian motion

$$
\begin{aligned}
&= \sum_{j=1}^n \mathbb{E}\big[(B_{t_j}-B_{t_{j-1}})^2(f(t_j)-f(t_{j-1}))^2\big]\\
&= \sum_{j=1}^n (f(t_j)-f(t_{j-1}))^2\,\mathbb{E}\big[(B_{t_j}-B_{t_{j-1}})^2\big]\\
&= \sum_{j=1}^n (t_j-t_{j-1})(f(t_j)-f(t_{j-1}))^2\\
&\leqslant 2\,|\Pi|\cdot\|f\|_\infty \sum_{j=1}^n |f(t_j)-f(t_{j-1})|
\end{aligned}
$$

$$\leqslant 2\,|\Pi|\cdot\|f\|_\infty \mathrm{VAR}_1(f;[0,T]) \xrightarrow[|\Pi|\to 0]{} 0$$

where we used the fact that a BV-function is necessarily bounded:

$$|f(t)| \leqslant |f(t)-f(0)| + |f(0)| \leqslant \mathrm{VAR}_1(f;[0,t]) + \mathrm{VAR}_1(f;\{0\}) \leqslant 2\mathrm{VAR}_1(f;[0,T])$$

for all $t \in [0,T]$.

Problem 14.12 (Solution) Replace, starting in the fourth line of the proof of Proposition 14.16, the argument as follows:

By the maximal inequalities (14.21) for Itô integrals we get

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\leqslant T}\left|\int_0^t \left[f(s)-f^\Pi(s)\right]dB_s\right|^2\right] \\
&\qquad \leqslant 4\int_0^T \mathbb{E}\left[\left|f(s)-f^\Pi(s)\right|^2\right]ds \\
&\qquad = 4\sum_{j=1}^n \int_{s_{j-1}}^{s_j} \mathbb{E}\left[|f(s)-f(s_{j-1})|^2\right]ds \\
&\qquad \leqslant 4\sum_{j=1}^n \int_{s_{j-1}}^{s_j} \underbrace{\sup_{u,v\in[s_{j-1},s_j]} \mathbb{E}\left[|f(u)-f(v)|^2\right]}_{\to 0,\ |\Pi|\to 0} ds \xrightarrow[|\Pi|\to 0]{} 0.
\end{aligned}$$

Problem 14.13 (Solution) To simplify notation, we drop the n in Π_n and write only $0 = t_0 < t_1 < \ldots < t_k = T$ and

$$\theta_{n,j}^\alpha = \theta_j = \alpha t_j + (1-\alpha)t_{j-1}.$$

We get

$$L_T(\alpha) := L^2(\mathbb{P})\text{-}\lim_{|\Pi|\to 0}\sum_{j=1}^k B_{\theta_j}(B_{t_j}-B_{t_{j-1}}) = \int_0^T B_s\,dB_s + \alpha T.$$

Indeed, we have

$$\begin{aligned}
&\sum_{j=1}^k B_{\theta_j}(B_{t_j}-B_{t_{j-1}}) \\
&\quad = \sum_{j=1}^k B_{t_{j-1}}(B_{t_j}-B_{t_{j-1}}) + \sum_{j=1}^k (B_{\theta_j}-B_{t_{j-1}})(B_{t_j}-B_{t_{j-1}}) \\
&\quad = \sum_{j=1}^k B_{t_{j-1}}(B_{t_j}-B_{t_{j-1}}) + \sum_{j=1}^k (B_{\theta_j}-B_{t_{j-1}})^2 + \sum_{j=1}^k (B_{t_j}-B_{\theta_j})(B_{\theta_j}-B_{t_{j-1}}) \\
&\quad = X + Y + Z.
\end{aligned}$$

We know already that $X \xrightarrow[|\Pi|\to 0]{L^2} \int_0^T B_s\,dB_s$. Moreover,

$$\begin{aligned}
\mathbb{V}\,Z &= \mathbb{V}\left(\sum_{j=1}^k (B_{t_j}-B_{\theta_j})(B_{\theta_j}-B_{t_{j-1}})\right) \\
&= \sum_{j=1}^k \mathbb{V}\left[(B_{t_j}-B_{\theta_j})(B_{\theta_j}-B_{t_{j-1}})\right]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{k} \mathbb{E}\left[(B_{t_j} - B_{\theta_j})^2 (B_{\theta_j} - B_{t_{j-1}})^2\right] \\
&= \sum_{j=1}^{k} \mathbb{E}\left[(B_{t_j} - B_{\theta_j})^2\right] \mathbb{E}\left[(B_{\theta_j} - B_{t_{j-1}})^2\right] \\
&= \sum_{j=1}^{k} (t_j - \theta_j)(\theta_j - t_{j-1}) \\
&= \alpha(1-\alpha) \sum_{j=1}^{k} (t_j - t_{j-1})(t_j - t_{j-1}) \xrightarrow{\text{as in Theorem 9.1}} 0.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\mathbb{E} Y = \mathbb{E}\left(\sum_{j=1}^{k} (B_{\theta_j} - B_{t_{j-1}})^2\right) &= \sum_{j=1}^{k} \mathbb{E}(B_{\theta_j} - B_{t_{j-1}})^2 \\
&= \sum_{j=1}^{k} (\theta_j - t_{j-1}) = \alpha \sum_{j=1}^{k} (t_j - t_{j-1}) = \alpha T.
\end{aligned}
$$

The L^2-convergence follows now *literally* as in the proof of Theorem 9.1.

Consequence: $L_T(\alpha) = \frac{1}{2}\left(B_T^2 + (2\alpha - 1)T\right)$, and this stochastic integral is a martingale if, and only if, $\alpha = 0$, i.e. if $\theta_j = t_{j-1}$ is the left endpoint of the interval.

For $\alpha = \frac{1}{2}$ we get the so-called Stratonovich or mid-point stochastic integral. This will obey the usual calculus rules (instead of Itô's rule). A first sign is the fact that

$$
L_T(\tfrac{1}{2}) = \tfrac{1}{2} B_T^2
$$

and we usually write

$$
L_T(\tfrac{1}{2}) = \int_0^T B_s \circ dB_s
$$

with the *Stratonovich-circle* $\circ$ to indicate the mid-point rule.

Problem 14.14 (Solution) a) Let τ_k be a sequence of stopping times with countably many, discrete values such that $\tau_k \downarrow \tau$. For example, $\tau_k := (\lfloor 2^k \tau \rfloor + 1)/2^k$, see Lemma A.15 in the appendix. Write $s_1 < \ldots < s_K$ for the values of τ_k. In particular,

$$
\mathbb{1}_{[0, T\wedge\tau_k)} = \sum_j \mathbb{1}_{\{T\wedge\tau_k = T\wedge s_j\}} \mathbb{1}_{[0, T\wedge s_j)}
$$

And so

$$
\{(s,\omega) : \mathbb{1}_{[0, T\wedge\tau_k(\omega))}(s) = 1\} = \bigcup_j [0, T\wedge s_j) \times \{T\wedge\tau_k = T\wedge s_j\}.
$$

Since $\{T\wedge\tau_k = T\wedge s_j\} \in \mathcal{F}_{T\wedge s_j}$, it is clear that

$$
\{(s,\omega) : \mathbb{1}_{[0, T\wedge\tau_k(\omega))}(s) = 1\} \cap ([0,t]\times\Omega) \in \mathcal{B}[0,t] \times \mathcal{F}_t \quad \text{for all} \quad t \geqslant 0
$$

and progressive measurability of $\mathbb{1}_{[0, T\wedge\tau_k)}$ follows.

b) Since $\mathsf{T} \wedge \tau_k \downarrow T \wedge \tau$ and $T \wedge \tau_k$ has only finitely many values, and we find

$$\lim_{k\to\infty} \mathbb{1}_{[0,T\wedge\tau_k)} = \mathbb{1}_{[0,T\wedge\tau]}$$

almost surely. Consequently, $\mathbb{1}_{[0,T\wedge\tau(\omega))]}(s)$ is also $\mathcal{P}$-measurable.

In fact, we do not need to prove the progressive measurability of $\mathbb{1}_{[0,T\wedge\tau)}$ to evaluate the integral. If you want to show it nevertheless, have a look at Problem 15.2 below.

c) Fix k and write $0 \leqslant s_1 < \ldots < s_K$ for the values of $T \wedge \tau_k$. Then

$$\begin{aligned}\int \mathbb{1}_{[0,T\wedge\tau_k)}(s)\,dB_s &= \int \sum_j \mathbb{1}_{[0,s_j)}(s)\mathbb{1}_{\{T\wedge\tau_k = T\wedge s_j\}}\,dB_s\\ &= \sum_j B_{s_j}\mathbb{1}_{\{T\wedge\tau_k = T\wedge s_j\}}\\ &= B_{T\wedge\tau_k}.\end{aligned}$$

d) $\underline{\mathbb{1}_{[0,T\wedge\tau)} = L^2\text{-}\lim_k \mathbb{1}_{[0,T\wedge\tau_k)}}$: This follows from

$$\begin{aligned}\mathbb{E}\int |\mathbb{1}_{[0,T\wedge\tau_k)}(s) - \mathbb{1}_{[0,T\wedge\tau)}(s)|^2\,ds &= \mathbb{E}\int |\mathbb{1}_{[T\wedge\tau,\,T\wedge\tau_k)}(s)|^2\,ds\\ &= \mathbb{E}\int \mathbb{1}_{[T\wedge\tau,\,T\wedge\tau_k)}(s)\,ds\\ &= \mathbb{E}(T\wedge\tau_k - T\wedge\tau) \xrightarrow[k\to\infty]{} 0\end{aligned}$$

by dominated convergence.

e) By the very definition of the stochastic integral we find now

$$\int \mathbb{1}_{[0,T\wedge\tau)}(s)\,dB_s \overset{\text{d)}}{=} L^2\text{-}\lim_k \int \mathbb{1}_{[0,T\wedge\tau_k)}(s)\,dB_s \overset{\text{c)}}{=} L^2\text{-}\lim_k B_{T\wedge\tau_k} = B_{T\wedge\tau}$$

by the continuity of Brownian motion and dominated convergence: $\sup_{s\leqslant T}|B_s|$ is integrable.

f) The result is, in the light of the localization principle of Theorem 14.13 not unexpected.

Problem 14.15 (Solution) Throughout the proof $t \geqslant 0$ is arbitrary but fixed.

- Clearly, $\varnothing, [0,T]\times\Omega \in \mathcal{P}$.
- Let $\Gamma \in \mathcal{P}$. Then

$$\Gamma^c \cap \big([0,t]\times\Omega\big) = \underbrace{\big([0,t]\times\Omega\big)}_{\in\mathcal{B}[0,t]\otimes\mathcal{F}_t} \smallsetminus \underbrace{\big(\Gamma\cap\big([0,t]\times\Omega\big)\big)}_{\in\mathcal{B}[0,t]\otimes\mathcal{F}_t} \in \mathcal{B}[0,t]\otimes\mathcal{F}_t,$$

thus $\Gamma^c \in \mathcal{P}$.

- Let $\Gamma_n \in \mathcal{P}$. By definition

$$\Gamma_n \cap \big([0,t]\times\Omega\big) \in \mathcal{B}[0,t]\otimes\mathcal{F}_t$$

and we can take the union over n to get

$$\Big(\bigcup_n \Gamma_n\Big) \cap \big([0,t]\times\Omega\big) = \bigcup_n \big(\Gamma_n \cap \big([0,t]\times\Omega\big)\big) \in \mathcal{B}[0,t]\otimes\mathcal{F}_t$$

i.e. $\bigcup_n \Gamma_n \in \mathcal{P}$.

Problem 14.16 (Solution) Let $f(t,\omega)$ be right-continuous on the interval $[0,T]$. (We consider only $T<\infty$ since the case of the infinite interval $[0,\infty)$ is actually easier.)

Set

$$f_n^T(s,\omega) := f\big(\tfrac{\lfloor 2^n s\rfloor+1}{2^n}\wedge T,\omega\big)$$

then

$$f_n^T(s,\omega) = \sum_k f\big(\tfrac{k+1}{2^n}\wedge T,\omega\big)\,\mathbb{1}_{[k2^{-n},(k+1)2^{-n})}(s) \quad (s\leqslant T)$$

and, since $(\lfloor 2^n s\rfloor+1)/2^n \downarrow s$, we find by right-continuity that $f_n\to f$ as $n\to\infty$. This means that it is enough to consider the $\mathcal{P}$-measurability of the step-function f_n.

Fix $n\geqslant 0$, write $t_j = j2^{-n}$. Then $t_0 = 0 < t_1 < \dots t_N \leqslant T$ for some suitable N. Observe that for any $x\in\mathbb{R}$

$$\{(s,\omega) : f(s,\omega)\leqslant x\} = \{T\}\times\{\omega : f(T,\omega)\leqslant x\} \cup \bigcup_{j=1}^N [t_{j-1},t_j)\times\{\omega : f(t_j,\omega)\leqslant x\}$$

and each set appearing in the union set on the right is in $\mathcal{B}[0,T]\otimes\mathcal{F}_T$.

This shows that f_n^T and f are $\mathcal{B}[0,T]\otimes\mathcal{F}_T$ measurable.

Now consider f_n^t and $f(t)\mathbb{1}_{[0,t]}$. We conclude, with the same reasoning, that both are $\mathcal{B}[0,t]\otimes\mathcal{F}_t$ measurable.

This shows that a right-continuous f is progressive.

If f is left-continuous, we use $\lfloor 2^n s\rfloor/2^n \uparrow s$ and define the approximating function as

$$g_n^T(s,\omega) = \sum_k f\big(\tfrac{k}{2^n}\wedge T,\omega\big)\,\mathbb{1}_{[k2^{-n},(k+1)2^{-n})}(s) \quad (s\leqslant T).$$

The rest of the proof is similar.

Problem 14.17 (Solution) By definition, there is a sequence f_n of elementary processes, i.e. of processes of the form

$$f_n(s,\omega) = \sum_j \phi_{j-1}(s)\mathbb{1}_{[t_{j-1},t_j)}(s)$$

where ϕ_{j-1} is $\mathcal{F}_{t_{j-1}}$ measurable such that $f_n\to f$ in $L^2(\mu_T\otimes\mathbb{P})$. In particular, there is a subsequence such that

$$\lim_{k\to\infty}\int_0^t |f_{n(k)}(s)|^2\,dA_s = \int_0^t |f(s)|^2\,dA_s \quad \text{a.s.}$$

so that it is enough to check that the integrals $\int_0^t |f_{n(j)}(s)|^2\,dA_s$ are adapted. By defintion

$$\int_0^t |f_{n(j)}(s)|^2\,dA_s = \sum_j \phi_{j-1}^2(A_{t_j\wedge t} - A_{t_{j-1}\wedge t})$$

and from this it is clear that the integral is $\mathcal{F}_t$ measurable for each t.

15 Stochastic Integrals: Beyond $\mathcal{L}^2_T$

Problem 15.1 (Solution) We know from the proof of Lemma 15.2 that for $f \in \mathcal{L}^2_T$ and any approximating sequence $(f_n)_{n\geqslant 0} \subset \mathcal{E}_T$ we have

$$\forall\, t \in [0,T] \quad \exists (n(j,t))_{j\geqslant 1} : \lim_{j\to\infty} \int_0^t |f_{n(j,t)}(s,\cdot)|^2\,ds = \int_0^t |f(s,\cdot)|^2\,ds \quad \text{almost surely.}$$

In particular the subsequence may depend on t. Since the rational numbers $\mathbb{Q}_+ \cap [0,T]$ are dense in $[0,T]$ we can construct, by a diagonal procedure, a subsequence $m(j)$ such that

$$\exists (m(j))_{j\geqslant 1} \quad \forall\, q \in [0,T]\cap\mathbb{Q} : \lim_{j\to\infty} \int_0^q |f_{m(j)}(s,\cdot)|^2\,ds = \int_0^q |f(s,\cdot)|^2\,ds \quad \text{almost surely.}$$

Observe that for any $t \in (0,T)$ there are rational numbers $q,r \in \mathbb{Q}\cap[0,T]$ such that $0 \leqslant r \leqslant t \leqslant q \leqslant T$. Then

$$\int_0^r |f_{m(j)}(s,\cdot)|^2\,ds \leqslant \int_0^t |f_{m(j)}(s,\cdot)|^2\,ds \leqslant \int_0^q |f_{m(j)}(s,\cdot)|^2\,ds$$

and

$$\begin{aligned}\varliminf_{j\to\infty} \int_0^r |f_{m(j)}(s,\cdot)|^2\,ds &\leqslant \varliminf_{j\to\infty} \int_0^t |f_{m(j)}(s,\cdot)|^2\,ds \\ &\leqslant \varlimsup_{j\to\infty} \int_0^t |f_{m(j)}(s,\cdot)|^2\,ds \leqslant \varlimsup_{j\to\infty} \int_0^q |f_{m(j)}(s,\cdot)|^2\,ds\end{aligned}$$

hence

$$\int_0^r |f(s,\cdot)|^2\,ds \leqslant \varliminf_{j\to\infty} \int_0^t |f_{m(j)}(s,\cdot)|^2\,ds \leqslant \varlimsup_{j\to\infty} \int_0^t |f_{m(j)}(s,\cdot)|^2\,ds \leqslant \int_0^q |f(s,\cdot)|^2\,ds.$$

Letting $r\uparrow t$ and $q \downarrow t$ along sequences of rational numbers, shows that

$$\int_0^t |f(s,\cdot)|^2\,ds \leqslant \varliminf_{j\to\infty} \int_0^t |f_{m(j)}(s,\cdot)|^2\,ds \leqslant \varlimsup_{j\to\infty} \int_0^t |f_{m(j)}(s,\cdot)|^2\,ds \leqslant \int_0^t |f(s,\cdot)|^2\,ds.$$

<u>Alternative Solution:</u> As in the proof of Lemma 15.2 there exists a sequence $(f_n)_{n\geqslant 0} \subset \mathcal{E}_T$ which converges to f in $L^2(\lambda_T \otimes \mathbb{P})$. There is a subsequence $(f_{n(j)})_{j\geqslant 0}$ such that

$$\int_0^T |f_{n(j)}(s,\cdot) - f(s,\cdot)|^2\,ds \to 0 \quad \text{almost surely.}$$

By the lower triangle inequality, we obtain

$$\left| \left(\int_0^t |f_{n(j)}(s,\cdot)|^2\,ds\right)^{\frac12} - \left(\int_0^t |f(s,\cdot)|^2\,ds\right)^{\frac12} \right| \leqslant \left(\int_0^t |f_{n(j)}(s,\cdot) - f(s,\cdot)|^2\,ds\right)^{\frac12}$$

$$\leqslant \left(\int_0^T |f_{n(j)}(s,\cdot) - f(s,\cdot)|^2\,ds\right)^{\frac{1}{2}}$$

$$\xrightarrow[j\to\infty]{} 0 \quad \text{almost surely}$$

for all $t \in [0,T]$.

Problem 15.2 (Solution) _Solution 1:_ We have that the process $t \mapsto \mathbb{1}_{[0,\tau(\omega))}(t)$ is adapted

$$\{\omega : \mathbb{1}_{[0,\tau(\omega))}(t) = 0\} = \{\tau \leqslant t\} \in \mathcal{F}_t$$

since τ is a stopping time. By Problem 14.16 we conclude that $\mathbb{1}_{[0,\tau)}$ is progressive.

Solution 2: Set $t_j = j2^{-n}$ and define

$$I_n^t(s,\omega) := \mathbb{1}_{[0,\tau(\omega))}\big(\tfrac{\lfloor 2^n s\rfloor}{2^n} \wedge t\big) = \sum_j \mathbb{1}_{[0,\tau(\omega))}(t_{j+1}\wedge t)\mathbb{1}_{[t_j,t_{j+1})}(s\wedge t).$$

Since $\lfloor 2^n s\rfloor/2^n \downarrow s$ we find, by right-continuity, $I_n^t \to \mathbb{1}_{[0,\tau)}$. Therefore, it is enough to check that I_n^t is $\mathcal{B}[0,t]\otimes\mathcal{F}_t$-measurable. But this is obvious from the form of I_n^t.

Problem 15.3 (Solution) Assume that σ_n are stopping times such that $(M_t^{\sigma_n}\mathbb{1}_{\{\sigma_n>0\}})_t$ is a martingale. Clearly,

- $\tau_n := \sigma_n \wedge n \uparrow \infty$ almost surely as $n\to\infty$;
- $\{\sigma_n > 0\} = \{\sigma_n \wedge n > 0\} = \{\tau_n > 0\}$;
- by optional stopping, the following process is a martingale for each n:

$$M_{t\wedge n}^{\sigma_n}\mathbb{1}_{\{\sigma_n>0\}} = M_t^{\sigma_n\wedge n}\mathbb{1}_{\{\sigma_n>0\}} = M_t^{\sigma_n\wedge n}\mathbb{1}_{\{\sigma_n\wedge n>0\}} = M_t^{\tau_n}\mathbb{1}_{\{\tau_n>0\}}.$$

Remark: This has an interesting consequence:

$$\mathbb{E}\left[\sup_{s\leqslant T}|M(s\wedge\tau_n)|^2\right] \overset{\text{Doob}}{\leqslant} 4\,\mathbb{E}\left[|M(\tau_n)|^2\right] \leqslant 4\,\mathbb{E}\left[|M(n)|^2\right].$$

16 Itô's Formula

Problem 16.1 (Solution) We try to identify the bits and pieces as parts of Itô's formula. For $f(x) = e^x$ we get $f'(x) = f''(x) = e^x$ and so

$$e^{B_t} - 1 = \int_0^t e^{B_s}\,dB_s + \frac{1}{2}\int_0^t e^{B_s}\,ds.$$

Thus,

$$X_t = e^{B_t} - 1 - \frac{1}{2}\int_0^t e^{B_s}\,ds.$$

With the same trick we try to find $f(x)$ such that $f'(x) = xe^{x^2}$. A moment's thought reveals that $f(x) = \frac{1}{2}e^{x^2}$ will do. Moreover $f''(x) = e^{x^2} + 2x^2e^{x^2}$. This then gives

$$\frac{1}{2}e^{B_t^2} - \frac{1}{2} = \int_0^t B_s e^{B_s^2}\,dB_s + \frac{1}{2}\int_0^t \left(e^{B_s^2} + 2B_s^2 e^{B_s^2}\right)ds$$

and we see that

$$Y_t = \frac{1}{2}\left(e^{B_t^2} - 1 - \int_0^t \left(e^{B_s^2} + 2B_s^2 e^{B_s^2}\right)ds\right).$$

Note: the integrand $B_s^2 e^{B_s^2}$ is not of class $\mathcal{L}_T^2$, thus we have to use a stopping technique (as in step 4^{o} of the proof of Itô's formula or as in Chapter 15).

Problem 16.2 (Solution) a) Set $F(x,y) = xy$ and $G(t) = (f(t), g(t))$.

Then $f(t)g(t) = F \circ G(t)$. If we differentiate this using the chain rule we get

$$\frac{d}{dt}(F \circ G) = \partial_x F \circ G(t) \cdot f'(t) + \partial_y F \circ G(t) \cdot g'(t) = g(t) \cdot f'(t) + f(t) \cdot g'(t)$$

(surprised?) and if we integrate this up we see

$$\begin{aligned} F \circ G(t) - F \circ G(0) &= \int_0^t f(s)g'(s)\,ds + \int_0^t g(s)f'(s)\,ds \\ &= \int_0^t f(s)\,dg(s) + \int_0^t g(s)\,df(s). \end{aligned}$$

Note: For the first equality we have to assume that f', g' exist Lebesgue a.e. and that their primitives are f and g, respectively. This is tantamount to saying that f, g are absolutely continuous with respect to Lebesgue measure.

b) $f(x,y) = xy$. Then $\partial_x f(x,y) = y, \partial_y f(x,y) = x$ and $\partial_x\partial_y f(x,y) = \partial_y\partial_x f(x,y) = 1$ and $\partial_x^2 f(x,y) = \partial_y^2 f(x,y) = 0$. Thus, the 2-dimensional Itô formula yields

$$b_t\beta_t = \int_0^t b_s\,d\beta_s + \int_0^t \beta_s\,db_s +$$

$$
\begin{aligned}
&\quad + \frac{1}{2}\int_0^t \partial_x^2 f(b_s,\beta_s)\,ds + \frac{1}{2}\int_0^t \partial_y^2 f(b_s,\beta_s)\,ds + \int_0^t \partial_x\partial_y f(b_s,\beta_s)\,d\langle b,\beta\rangle_s \\
&= \int_0^t b_s\,d\beta_s + \int_0^t \beta_s\,db_s + \langle b,\beta\rangle_t.
\end{aligned}
$$

If $b \perp\!\!\!\perp \beta$ we have $\langle b,\beta\rangle \equiv 0$ (note our Itô formula has no mixed second derivatives!) and we get the formula as in the statement. Otherwise we have to take care of $\langle b,\beta\rangle$. This is not so easy to calculate since we need more information on the joint distribution. In general, we have

$$
\langle b,\beta\rangle_t = \lim_{|\Pi|\to 0} \sum_{t_j,t_{j-1}} (b(t_j) - b(t_{j-1}))(\beta(t_j) - \beta(t_{j-1})).
$$

Where Π stands for a partition of the interval $[0,t]$.

Problem 16.3 (Solution) Consider the two-dimensional Itô process $X_t = (t, B_t)$ with parameters

$$
\sigma \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad b \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
$$

Applying the Itô formula (16.8) we get

$$
\begin{aligned}
f(t,B_t) - f(0,0) &= f(X_t) - f(X_0) \\
&= \int_0^t \Big(\partial_1 f(X_s)\sigma_{11} + \partial_2 f(X_s)\sigma_{21}\Big)\,dB_s \\
&\quad + \int_0^t \Big(\partial_1 f(X_s) b_1 + \partial_2 f(X_s) b_2 + \frac{1}{2}\partial_2\partial_2 f(X_s)\sigma_{21}^2\Big)\,ds \\
&= \int_0^t \partial_2 f(X_s)\,dB_s + \int_0^t \Big(\partial_1 f(X_s) b_1 + \frac{1}{2}\partial_2\partial_2 f(X_s)\Big)\,ds \\
&= \int_0^t \frac{\partial f}{\partial x}(s,B_s)\,dB_s + \int_0^t \left(\frac{\partial f}{\partial t}(s,B_s) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s,B_s)\right)ds.
\end{aligned}
$$

In the same way we obtain the d-dimensional counterpart:

Let $(B_t^1,\dots,B_t^d)_{t\geqslant 0}$ be a BM^d and $f:[0,\infty)\times\mathbb{R}^d \to \mathbb{R}$ be a function of class $\mathcal{C}^{1,2}$. Consider the $d+1$-dimensional Itô process $X_t = (t, B_t^1,\dots,B_t^d)$ with parameters

$$
\sigma \in \mathbb{R}^{d+1\times d}, \quad \sigma_{ik} = \begin{cases} 1, & \text{if } i = k+1; \\ 0, & \text{else;} \end{cases} \qquad \text{and} \quad b = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.
$$

The multidimensional Itô formula (16.8) yields

$$
\begin{aligned}
&f(t,B_t^1,\dots,B_t^d) - f(0,0,\dots,0) \\
&= f(X_t) - f(X_0) \\
&= \sum_{k=1}^d \int_0^t \left[\sum_{j=1}^{d+1} \partial_j f(X_s)\sigma_{jk}\right] dB_s^k + \sum_{j=1}^{d+1}\int_0^t \partial_j f(X_s) b_j\,ds + \frac{1}{2}\sum_{i,j=1}^{d+1}\int_0^t \partial_i\partial_j f(X_s)\sum_{k=1}^d \sigma_{ik}\sigma_{jk}\,ds
\end{aligned}
$$

$$= \sum_{k=1}^{d} \int_0^t \partial_{k+1} f(X_s)\, dB_s^k + \int_0^t \partial_1 f(X_s)\, ds + \frac{1}{2} \sum_{j=2}^{d+1} \int_0^t \partial_j \partial_j f(X_s)\, ds$$

$$= \sum_{k=1}^{d} \int_0^t \frac{\partial f}{\partial x_k}(s, B_s^1, \dots, B_s^d)\, dB_s^k + \int_0^t \left(\frac{\partial f}{\partial t}(s, B_s^1, \dots, B_s^d) + \frac{1}{2} \sum_{k=1}^{d} \frac{\partial^2 f}{\partial x_k^2}(s, B_s^1, \dots, B_s^d) \right) ds.$$

Problem 16.4 (Solution) Let $B_t = (B_t^1, \dots, B_t^d)$ be a BM^d and $f \in \mathcal{C}^{1,2}((0,\infty) \times \mathbb{R}^d, \mathbb{R})$ as in Theorem 5.6. Then the multidimensional time-dependent Itô's formula shown in Problem 16.3 yields

$$\begin{aligned} M_t^f &= f(t, B_t) - f(0, B_0) - \int_0^t Lf(s, B_s)\, ds \\ &= f(t, B_t) - f(0, B_0) - \int_0^t \left(\frac{\partial}{\partial t} f(s, B_s) + \frac{1}{2} \Delta_x f(s, B_s) \right) ds \\ &= \sum_{k=1}^{d} \int_0^t \frac{\partial f}{\partial x_k}(s, B_s^1, \dots, B_s^d)\, dB_s^k. \end{aligned}$$

By Theorem 14.13 it follows that M_t^f is a martingale (note that the assumption (5.5) guarantees that the integrand is of class $\mathcal{L}_T^2$!)

Problem 16.5 (Solution) First we show that $X_t = e^{t/2} \cos B_t$ is a martingale. We use the time-dependent Itô's formula from Problem 16.3. Therefore, we set $f(t,x) = e^{t/2} \cos x$. Then

$$\frac{\partial f}{\partial t}(t,x) = \frac{1}{2} e^{t/2} \cos x, \qquad \frac{\partial f}{\partial x}(t,x) = -e^{t/2} \sin x, \qquad \frac{\partial^2 f}{\partial x^2}(t,x) = -e^{t/2} \cos x.$$

Hence we obtain

$$\begin{aligned} X_t = e^{t/2} \cos B_t &= f(t, B_t) - f(0,0) + 1 \\ &= \int_0^t \frac{\partial f}{\partial x}(s, B_s)\, dB_s + \int_0^t \left(\frac{\partial f}{\partial t}(s, B_s) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(s, B_s) \right) ds + 1 \\ &= -\int_0^t e^{s/2} \sin B_s\, dB_s + \int_0^t \left(\frac{1}{2} e^{s/2} \cos B_s - \frac{1}{2} e^{s/2} \cos B_s \right) ds + 1 \\ &= -\int_0^t e^{s/2} \sin B_s\, dB_s + 1, \end{aligned}$$

and the claim follows from Theorem 14.13.

Analogously, we show that $Y_t = (B_t + t) e^{-B_t - t/2}$ is a martingale. We set $f(t,x) = (x + t) e^{-x-t/2}$. Then

$$\begin{aligned} \frac{\partial f}{\partial t}(t,x) &= e^{-x-t/2} - \frac{1}{2}(x+t) e^{-x-t/2}, \\ \frac{\partial f}{\partial x}(t,x) &= e^{-x-t/2} - (x+t) e^{-x-t/2}, \\ \frac{\partial f}{\partial x^2}(t,x) &= -2e^{-x-t/2} + (x+t) e^{-x-t/2}. \end{aligned}$$

By the time-dependent Itô's formula we have

$$\begin{aligned} Y_t &= (B_t + t) e^{-B_t - t/2} \\ &= f(t, B_t) - f(0,0) \end{aligned}$$

$$
\begin{aligned}
&= \int_0^t \left(e^{-B_s - s/2} - (B_s + s)e^{-B_s - s/2}\right) dB_s + \\
&\quad + \int_0^t \left(e^{-B_s - s/2} - \frac{1}{2}(B_s + s)e^{-B_s - s/2} + \frac{1}{2}(-2e^{-B_s - s/2} + (B_s + s)e^{-B_s - s/2})\right) ds \\
&= \int_0^t \left(e^{-B_s - s/2} - (B_s + s)e^{-B_s - s/2}\right) dB_s.
\end{aligned}
$$

Again, from Theorem 14.13 we deduce that Y_t is a martingale.

Problem 16.6 (Solution) a) The stochastic integrals exist if b_s/r_s and β_s/r_s are in $\mathcal{L}^2_T$. As $|b_s/r_s| \leqslant 1$ we get

$$
\|b/r\|^2_{L^2(\lambda_T \otimes \mathbb{P})} = \int_0^T \left[\mathbb{E}\left(|b_s/r_s|^2\right)\right] ds \leqslant \int_0^T 1 ds = T < \infty.
$$

Since b_s/r_s is adapted and has continuous sample paths, it is progressive and so an element of $\mathcal{L}^2_T$. Analogously, $|\beta_s/r_s| \leqslant 1$ implies $\beta_s/r_s \in \mathcal{L}^2_T$.

b) We use Lévy's characterization of a BM^1, Theorem 9.12 or 17.5. From Theorem 14.13 it follows that

- $t \mapsto \int_0^t b_s/r_s \, db_s$, $t \mapsto \int_0^t \beta_s/r_s \, d\beta_s$ are continuous; thus $t \mapsto W_t$ is a continuous process.
- $\int_0^t b_s/r_s \, db_s$, $\int_0^t \beta_s/r_s \, d\beta_s$ are square integrable martingales, and so is W_t.
- the quadratic variation is given by

$$
\begin{aligned}
\langle W \rangle_t &= \langle b/r \bullet b \rangle_t + \langle \beta/r \bullet \beta \rangle_t \\
&= \int_0^t b_s^2/r_s^2 \, ds + \int_0^t \beta_s^2/r_s^2 \, ds \\
&= \int_0^t \frac{b_s^2 + \beta_s^2}{r_s^2} \, ds \\
&= \int_0^t ds = t,
\end{aligned}
$$

  i.e. $(W_t^2 - t)_{t \geqslant 0}$ is a martingale.

Therefore, W_t is a BM^1.

Note, that the above processes can be used to calculate *Lévy's stochastic area formula*, see Protter [7, Chapter II, Theorem 43]

Problem 16.7 (Solution) The function $f = u + iv$ is analytic, and as such it satisfies the Cauchy–Riemann equations, see e.g. Rudin [10, Theorem 11.2],

$$
u_x = v_y \quad \text{and} \quad u_y = -v_x.
$$

First, we show that $u(b_t, \beta_t)$ is a BM^1. Therefore we apply Itô's formula

$$
\begin{aligned}
&u(b_t, \beta_t) - u(b_0, \beta_0) \\
&= \int_0^t u_x(b_s, \beta_s) \, db_s + \int_0^t u_y(b_s, \beta_s) \, d\beta_s + \frac{1}{2} \int_0^t \left(u_{xx}(b_s, \beta_s) + u_{yy}(b_s, \beta_s)\right) ds
\end{aligned}
$$

$$= \int_0^t u_x(b_s, \beta_s)\, db_s + \int_0^t u_y(b_s, \beta_s)\, d\beta_s,$$

where the last term cancels as $u_{xx} = v_{yx}$ and $u_{yy} = -v_{xy}$. Theorem 14.13 implies

- $t \mapsto u(b_t, \beta_t) = \int_0^t u_x(b_s, \beta_s)\, db_s + \int_0^t u_y(b_s, \beta_s)\, d\beta_s$ is a continuous process.
- $\int_0^t u_x(b_s, \beta_s)\, db_s$, $\int_0^t u_y(b_s, \beta_s)\, d\beta_s$ are square integrable martingales, and so $u(b_t, \beta_t)$ is a square integrable martingale.
- the quadratic variation is given by

$$\begin{aligned}\langle u(b, \beta)\rangle_t &= \langle u_x(b, \beta) \bullet b\rangle_t + \langle u_y(b, \beta) \bullet \beta\rangle_t \\ &= \int_0^t u_x^2(b_s, \beta_s)\, ds + \int_0^t u_y^2(b_s, \beta_s)\, ds = \int_0^t 1\, ds = t,\end{aligned}$$

  i.e. $(u^2(b_t, \beta_t) - t)_{t\geqslant 0}$ is a martingale.

Due to Lévy's characterization of a BM^1, Theorem 9.12 or 17.5, we know that $u(b_t, \beta_t)$ is a BM^1. Analogously, we see that $v(b_t, \beta_t)$ is also a BM^1. Just note that, due to the Cauchy–Riemann equations we get from $u_x^2 + u_y^2 = 1$ also $v_y^2 + v_x^2 = 1$.

The quadratic covariation is (we drop the arguments, for brevity):

$$\begin{aligned}\langle u, v\rangle_t &= \frac{1}{4}\left(\langle u + v\rangle_t - \langle u - v\rangle_t\right) \\ &= \frac{1}{4}\left(\int_0^t (u_x + v_x)^2\, ds + \int_0^t (u_y + v_y)^2\, ds - \int_0^t (u_x - v_x)^2\, ds - \int_0^t (u_y - v_y)^2\, ds\right) \\ &= \int_0^t (u_x v_x + u_y v_y)\, ds \\ &= \int_0^t (-v_y u_y + u_y v_y)\, ds = 0.\end{aligned}$$

As an abbreviation we write $u_t = u(b_t, \beta_t)$ and $v_t = v(b_t, \beta_t)$. Applying Itô's formula to the function $g(u_t, v_t) = e^{i(\xi u_t + \eta v_t)}$ and $s < t$ yields

$$g(u_t, v_t) - g(u_s, v_s) = i\xi \int_s^t g(u_r, v_r)\, du_r + i\eta \int_s^t g(u_r, v_r)\, dv_r - \frac{1}{2}(\xi^2 + \eta^2) \int_s^t g(u_r, v_r)\, dr,$$

as the quadratic covariation $\langle u, v\rangle_t = 0$. Since $|g| \leqslant 1$ and since $g(u_t, v_t)$ is progressive, the integrand is in $\mathcal{L}_T^2$ and the above stochastic integrals exist. From Theorem 14.13 we deduce that

$$\mathbb{E}\left(\int_s^t g(u_r, v_r)\, du_r\, \mathbb{1}_F\right) = 0 \quad\text{and}\quad \mathbb{E}\left(\int_s^t g(u_r, v_r)\, dv_r\, \mathbb{1}_F\right) = 0.$$

for all $F \in \sigma(u_r, v_r : r \leqslant s) =: \mathcal{F}_s$. If we multiply the above equality by $e^{-i(\xi u_s + \eta v_s)}\mathbb{1}_F$ and take expectations, we get

$$\underbrace{\mathbb{E}\left(g(u_t - u_s, v_t - v_s)\mathbb{1}_F\right)}_{=\Phi(t)} = \mathbb{P}(F) - \frac{1}{2}(\xi^2 + \eta^2)\int_0^t \underbrace{\mathbb{E}\left(g(u_r - u_s, v_r - v_s)\mathbb{1}_F\right)}_{=\Phi(r)}\, dr.$$

Since this integral equation has a unique solution (use Gronwall's lemma, Theorem A.43), we get

$$\mathbb{E}(e^{i(\xi(u_t - u_s) + \eta(v_t - v_s))}\mathbb{1}_F) = \mathbb{P}(F)\, e^{-\frac{1}{2}(t-s)(\xi^2 + \eta^2)}$$

$$
\begin{aligned}
&= \mathbb{P}(F)\, e^{-\frac{1}{2}(t-s)\xi^2} e^{-\frac{1}{2}(t-s)\eta^2} \\
&= \mathbb{P}(F)\ \mathbb{E}(e^{i\xi(u_t-u_s)})\, \mathbb{E}(e^{i\eta(v_t-v_s)}).
\end{aligned}
$$

From this we deduce with Lemma 5.4 that $(u(b_t,\beta_t), v(b_t,\beta_t))$ is a BM^2.

Note that the above calculation is essentially the proof of Lévy's characterization theorem. Only a few modifications are necessary for the proof of the multidimensional version, see e.g. Karatzas, Shreve [5, Theorem 3.3.16].

Problem 16.8 (Solution) Let $X_t = \int_0^t \sigma(s)\,dB_s + \int_0^t b(s)\,ds$ be an d-dimensional Itô process. Assuming that $f = u + iv$ and thus $u = \operatorname{Re} f = \frac{1}{2} f + \frac{1}{2}\bar f$ and $v = \operatorname{Im} f = \frac{1}{2i} f + \frac{1}{2i}\bar f$ are $\mathcal{C}^2$-functions, we may apply the real d-dimensional Itô formula (16.9) to the functions $u, v : \mathbb{R}^d \to \mathbb{R}$,

$$
\begin{aligned}
&f(X_t) - f(X_0) \\
&= u(X_t) - u(X_0) + i\big(v(X_t) - v(X_0)\big) \\
&= \int_0^t \nabla u(X_s)^\top \sigma(s)\,dB_s + \int_0^t \nabla u(X_s)^\top b(s)\,ds + \frac{1}{2}\int_0^t \operatorname{trace}\big(\sigma(s)^\top D^2 u(X_s)\sigma(s)\big)\,ds \\
&\quad + i\Big(\int_0^t \nabla v(X_s)^\top \sigma(s)\,dB_s + \int_0^t \nabla v(X_s)^\top b(s)\,ds + \frac{1}{2}\int_0^t \operatorname{trace}\big(\sigma(s)^\top D^2 v(X_s)\sigma(s)\big)\,ds\Big) \\
&= \int_0^t \nabla f(X_s)^\top \sigma(s)\,dB_s + \int_0^t \nabla f(X_s)^\top b(s)\,ds + \frac{1}{2}\int_0^t \operatorname{trace}\big(\sigma(s)^\top D^2 f(X_s)\sigma(s)\big)\,ds,
\end{aligned}
$$

by the linearity of the differential operators and the (stochastic) integral.

Problem 16.9 (Solution) a) By definition we have $\operatorname{supp}\chi \subset [-1,1]$ hence it is obvious that for $\chi_n(x) := n\chi(nx)$ we have $\operatorname{supp}\chi_n \subset [-1/n, 1/n]$. Substituting $y = nx$ we get

$$
\int_{-1/n}^{1/n} \chi_n(x)\,dx = \int_{-1/n}^{1/n} n\chi(nx)\,dx = \int_{-1}^{1} \chi(y)\,dy = 1
$$

b) For derivatives of convolutions we know that $\partial(f \star \chi_n) = f \star (\partial\chi_n)$. Hence we obtain

$$
\begin{aligned}
|\partial^k f_n(x)| &= |f \star (\partial^k \chi_n)(x)| \\
&= \Big|\int_{\mathbb{B}(x,1/n)} f(y)\partial^k \chi_n(x-y)\,dy\Big| \\
&\leqslant \sup_{y\in\mathbb{B}(x,1/n)} |f(y)| \int_{\mathbb{R}} n\,|\partial^k \chi(n(x-y))|\,dy \\
&= \sup_{y\in\mathbb{B}(x,1/n)} |f(y)| \int_{\mathbb{R}} n^k\,|\partial^k \chi(z)|\,dz \\
&= \sup_{y\in\mathbb{B}(x,1/n)} |f(y)|\, n^k\, \|\partial^k \chi\|_{L^1},
\end{aligned}
$$

where we substituted $z = n(y-x)$ in the penultimate step.

d) For $x \in \mathbb{R}$ we have

$$
|f \star \chi_n(x) - f(x)| = \Big|\int_{\mathbb{R}} (f(y) - f(x))\chi_n(x-y)\,dy\Big|
$$

$$\leqslant \Big| \sup_{y\in\mathbb{B}(x,1/n)} |f(y)-f(x)| \cdot \|\chi\|_{L^1}$$

$$= \sup_{y\in\mathbb{B}(x,1/n)} |f(y)-f(x)|.$$

This shows that $\lim_{n\to\infty}|f\star\chi_n(x)-f(x)|=0$, i.e. $\lim_{n\to\infty} f\star\chi_n(x)=f(x)$, at all x where f is continuous.

c) Using the above result and taking the supremum over all $x\in\mathbb{R}$ we get

$$\sup_{x\in\mathbb{R}}|f\star\chi_n(x)-f(x)| \leqslant \sup_{x\in\mathbb{R}}\sup_{y\in\mathbb{B}(x,1/n)}|f(y)-f(x)|.$$

Thus $\lim_{n\to\infty}\|f\star\chi_n-f\|_\infty=0$ whenever the function f is uniformly continuous.

Problem 16.10 (Solution) We follow the hint and use Lévy's characterization of a BM^1, Theorem 9.12 or 17.5.

- $t\mapsto\beta_t$ is a continuous process.
- the integrand $\operatorname{sgn} B_s$ is bounded, hence it is in $\mathcal{L}^2_T$ for any $T>0$.
- by Theorem 14.13 β_t is a square integrable martingale
- by Theorem 14.13 the quadratic variation is given by

$$\langle\beta\rangle_t = \left\langle \int_0^\bullet \operatorname{sgn}(B_s)\,dB_s \right\rangle_t = \int_0^t (\operatorname{sgn}(B_s))^2\,ds = \int_0^t ds = t$$

  i.e. $(\beta_t^2-t)_{t\geqslant0}$ is also a martingale.

Thus, β is a BM^1.

17 Applications of Itô's Formula

Problem 17.1 (Solution) **Lemma.** *Let $(B_t, \mathcal{F}_t)_{t\geqslant 0}$ be a* BM^d*, $f = (f_1, \dots, f_d)$, $f_j \in L^2_{\mathcal{P}}(\lambda_T \otimes \mathbb{P})$ for all $T > 0$, and assume that $|f_j(s,\omega)| \leqslant C$ for some $C > 0$ and all $s \geqslant 0, 1 \leqslant j \leqslant d$, and $\omega \in \Omega$. Then*

$$\exp\left(\sum_{j=1}^d \int_0^t f_j(s)\,dB_s^j - \frac{1}{2}\sum_{j=1}^d \int_0^t f_j^2(s)\,ds\right), \quad t \geqslant 0, \tag{17.1}$$

is a martingale for the filtration $(\mathcal{F}_t)_{t\geqslant 0}$.

Proof. Set $X_t = \sum_{j=1}^d \int_0^t f_j(s)\,dB_s^j - \frac{1}{2}\sum_{j=1}^d \int_0^t f_j^2(s)\,ds$. Itô's formula, Theorem 16.5, yields

$$\begin{aligned}
e^{X_t} - 1 &= \sum_{j=1}^d \int_0^t e^{X_s} f_j(s)\,dB_s^j - \frac{1}{2}\sum_{j=1}^d \int_0^t e^{X_s} f_j^2(s)\,ds + \frac{1}{2}\sum_{j=1}^d \int_0^t e^{X_s} f_j^2(s)\,ds \\
&= \sum_{j=1}^d \int_0^t \exp\left(\sum_{k=1}^d \int_0^s f_k(r)\,dB_r^k - \frac{1}{2}\sum_{k=1}^d \int_0^s f_k^2(r)\,dr\right) f_j(s)\,dB_s^j \\
&= \sum_{j=1}^d \int_0^t \prod_{k=1}^d \exp\left(\int_0^s f_k(r)\,dB_r^k - \frac{1}{2}\int_0^s f_k^2(r)\,dr\right) f_j(s)\,dB_s^j.
\end{aligned}$$

If we can show that the integrand is in $L^2_{\mathcal{P}}(\lambda_T \otimes \mathbb{P})$ for every $T > 0$, then Theorem 14.13 applies and shows that the stochastic integral, hence e^{X_t}, is a martingale.

We will see that we can reduce the d-dimensional setting to a one-dimensional setting. The essential step in the proof is the analogue of the estimate on page 250, line 6 from above. In the d-dimensional setting we have for each $k = 1, \dots, d$

$$\begin{aligned}
\mathbb{E}\left[\left|e^{\sum_{j=1}^d \int_0^T f_j(r)\,dB_r^j - \frac{1}{2}\sum_{j=1}^d \int_0^T f_j^2(r)\,dr} f_k(T)\right|^2\right] &\leqslant C^2\, \mathbb{E}\left[e^{2\sum_{j=1}^d \int_0^T f_j(r)\,dB_r^j}\right] \\
&= C^2\, \mathbb{E}\left[\prod_{j=1}^d e^{2\int_0^T f_j(r)\,dB_r^j}\right] \\
&\leqslant C^2 \prod_{j=1}^d \left(\mathbb{E}\left[e^{2d\int_0^T f_j(r)\,dB_r^j}\right]\right)^{1/d}.
\end{aligned}$$

In the last step we used the generalized Hölder inequality

$$\int \prod_{k=1}^n \phi_k\,d\mu \leqslant \prod_{k=1}^n \left(\int |\phi_k|^{p_k}\,d\mu\right)^{1/p_k} \qquad \forall (p_1, \dots, p_n) \in [1,\infty)^n : \textstyle\sum_{k=1}^n \frac{1}{p_k} = 1$$

with $n = d$ and $p_1 = \ldots = p_d = d$. Now the one-dimensional argument with df_j playing the role of f shows (cf. page 250, line 9 from above)

$$\mathbb{E}\left[\left|e^{\sum_{j=1}^d \int_0^T f_j(r)\,dB_r^j - \frac{1}{2}\sum_{j=1}^d \int_0^T f_j^2(r)\,dr} f_k(T)\right|^2\right] \leqslant C^2 \prod_{j=1}^d \left(\mathbb{E}\left[e^{2d\int_0^T f_j(r)\,dB_r^j}\right]\right)^{1/d}$$

$$\leqslant C^2 e^{2dC^2T} < \infty. \qquad \square$$

Problem 17.2 (Solution) As for a Brownian motion one can see that the independent increments property of a Poisson process is equivalent to saying that $N_t - N_s \perp\!\!\!\perp \mathcal{F}_s^N$ for all $s \leqslant t$, cf. Lemma 2.10 or Section 5.1. Thus, we have for $s \leqslant t$

$$\begin{aligned}
\mathbb{E}(N_t - t \,|\, \mathcal{F}_s^N) &= \mathbb{E}(N_t - N_s - (t-s) \,|\, \mathcal{F}_s^N) + \mathbb{E}(N_s - s \,|\, \mathcal{F}_s^N) \\
&\underset{\text{pull out}}{\overset{N_t - N_s \perp\!\!\!\perp \mathcal{F}_s^N}{=}} \mathbb{E}(N_t - N_s - (t-s)) + N_s - s \\
&\overset{N_t - N_s \sim N_{t-s}}{=} \mathbb{E}(N_t - N_s) - (t-s) + N_s - s \\
&= \mathbb{E}(N_{t-s}) - (t-s) + N_s - s \\
&= N_s - s.
\end{aligned}$$

Observe that

$$\begin{aligned}
(N_t - t)^2 - t &= \big(N_t - N_s - (t-s) + (N_s - s)\big)^2 - t \\
&= \big(N_t - N_s - (t-s)\big)^2 + (N_s - s)^2 + 2(N_s - s)(N_t - N_s - t + s) - t.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\big((N_t - t)^2 - t\big) - \big((N_s - s)^2 - s\big) \\
&\qquad = \big(N_t - N_s - (t-s)\big)^2 + 2(N_s - s)(N_t - N_s - t + s) - (t-s).
\end{aligned}$$

Now take $\mathbb{E}(\cdots \,|\, \mathcal{F}_s^N)$ in the last equality and observe that $N_t - N_s \perp\!\!\!\perp \mathcal{F}_s$. Then

$$\begin{aligned}
&\mathbb{E}\Big[\big((N_t - t)^2 - t\big) - \big((N_s - s)^2 - s\big) \,\Big|\, \mathcal{F}_s^N\Big] \\
&\overset{N_t - N_s \perp\!\!\!\perp \mathcal{F}_s^N}{=} \mathbb{E}\Big[\big(N_t - N_s - (t-s)\big)^2\Big] + 2\,\mathbb{E}\Big[(N_s - s)(N_t - N_s - t + s) \,\Big|\, \mathcal{F}_s^N\Big] - (t-s) \\
&\underset{\text{pull out}}{\overset{N_t - N_s \sim N_{t-s}}{=}} \mathbb{E}\Big[\big(N_{t-s} - (t-s)\big)^2\Big] + 2(N_s - s)\,\mathbb{E}\Big[(N_t - N_s - t + s) \,\Big|\, \mathcal{F}_s^N\Big] - (t-s) \\
&\overset{N_t - N_s \perp\!\!\!\perp \mathcal{F}_s^N}{=} \mathbb{V}\, N_{t-s} + 2(N_s - s)\,\mathbb{E}(N_t - N_s - t + s) - (t-s) \\
&= t - s + 2(N_s - s)\cdot 0 - (t-s) = 0.
\end{aligned}$$

Since $t \mapsto N_t$ is not continuous, this does not contradict Theorem 17.5.

Problem 17.3 (Solution) _Solution 1:_ Note that

$$\begin{aligned}
\mathbb{Q}(W(t_j) \in A_j, \forall j = 1, \dots, n) &= \int \prod_{j=1}^n \mathbb{1}_{A_j}(W(t_j))\, d\mathbb{Q} \\
&= \int \prod_{j=1}^n \mathbb{1}_{A_j}(B(t_j) - \xi t_j)\, e^{\xi B(T) - \frac12 \xi^2 T}\, d\mathbb{P}.
\end{aligned}$$

By the tower property and the fact that $e^{\xi B(t) - \frac12 \xi^2 t}$ is a martingale we get

$$\int \prod_{j=1}^n \mathbb{1}_{A_j}(B(t_j) - \xi t_j)\, e^{\xi B(T) - \frac12 \xi^2 T}\, d\mathbb{P}$$

$$
\begin{aligned}
&= \mathbb{E}\left[\mathbb{E}\left(\prod_{j=1}^{n} \mathbb{1}_{A_j}(B(t_j) - \xi t_j)\, e^{\xi B(T) - \frac{1}{2}\xi^2 T} \,\middle|\, \mathcal{F}_{t_n}\right)\right] \\
&= \mathbb{E}\left[\prod_{j=1}^{n} \mathbb{1}_{A_j}(B(t_j) - \xi t_j)\, \mathbb{E}\left(e^{\xi B(T) - \frac{1}{2}\xi^2 T} \,\middle|\, \mathcal{F}_{t_n}\right)\right] \\
&= \mathbb{E}\left[\prod_{j=1}^{n} \mathbb{1}_{A_j}(B(t_j) - \xi t_j)\, e^{\xi B(t_n) - \frac{1}{2}\xi^2 t_n}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left(\prod_{j=1}^{n} \mathbb{1}_{A_j}(B(t_j) - \xi t_j)\, e^{\xi B(t_n) - \frac{1}{2}\xi^2 t_n} \,\middle|\, \mathcal{F}_{t_{n-1}}\right)\right] \\
&= \mathbb{E}\left[\prod_{j=1}^{n-1} \mathbb{1}_{A_j}(B(t_j) - \xi t_j)\, e^{\xi B(t_{n-1}) - \frac{1}{2}\xi^2 t_{n-1}} \times\right. \\
&\qquad \left. \times\, \mathbb{E}\left(\mathbb{1}_{A_n}(B(t_n) - \xi t_n)\, e^{\xi (B(t_n) - B(t_{n-1})) - \frac{1}{2}\xi^2 (t_n - t_{n-1})} \,\middle|\, \mathcal{F}_{t_{n-1}}\right)\right]
\end{aligned}
$$

Now, since $B(t_n) - B(t_{n-1}) \perp\!\!\!\perp \mathcal{F}_{t_{n-1}}$ we get

$$
\begin{aligned}
&\mathbb{E}\left(\mathbb{1}_{A_n}(B(t_n) - \xi t_n)\, e^{\xi (B(t_n) - B(t_{n-1})) - \frac{1}{2}\xi^2 (t_n - t_{n-1})} \,\middle|\, \mathcal{F}_{t_{n-1}}\right) \\
&= \mathbb{E}\left(\mathbb{1}_{A_n}((B(t_n) - B(t_{n-1})) - \xi(t_n - t_{n-1}) + B(t_{n-1}) - \xi t_{n-1}) \times\right. \\
&\qquad \left. \times\, e^{\xi (B(t_n) - B(t_{n-1})) - \frac{1}{2}\xi^2 (t_n - t_{n-1})} \,\middle|\, \mathcal{F}_{t_{n-1}}\right) \\
&= \mathbb{E}\left(\mathbb{1}_{A_n}((B(t_n) - B(t_{n-1})) - \xi(t_n - t_{n-1}) + y) \times\right. \\
&\qquad \left. \times\, e^{\xi (B(t_n) - B(t_{n-1})) - \frac{1}{2}\xi^2 (t_n - t_{n-1})}\right)\Bigg|_{y = B(t_{n-1}) - \xi t_{n-1}}
\end{aligned}
$$

A direct calculation now gives

$$
\begin{aligned}
&\mathbb{E}\left(\mathbb{1}_{A_n}((B(t_n) - B(t_{n-1})) - \xi(t_n - t_{n-1}) + y) e^{\xi (B(t_n) - B(t_{n-1})) - \frac{1}{2}\xi^2 (t_n - t_{n-1})}\right) \\
&= \mathbb{E}\left(\mathbb{1}_{A_n}(B(t_n - t_{n-1}) - \xi(t_n - t_{n-1}) + y) e^{\xi B(t_n - t_{n-1}) - \frac{1}{2}\xi^2 (t_n - t_{n-1})}\right) \\
&= \frac{1}{\sqrt{2\pi(t_n - t_{n-1})}} \int \mathbb{1}_{A_n}(x - \xi(t_n - t_{n-1}) + y) e^{\xi x - \frac{1}{2}\xi^2 (t_n - t_{n-1})}\, e^{-\frac{1}{2(t_n - t_{n-1})} x^2}\, dx \\
&= \frac{1}{\sqrt{2\pi(t_n - t_{n-1})}} \int \mathbb{1}_{A_n}(x - \xi(t_n - t_{n-1}) + y) e^{-\frac{1}{2(t_n - t_{n-1})}(x - \xi(t_n - t_{n-1}))^2}\, dx \\
&= \frac{1}{\sqrt{2\pi(t_n - t_{n-1})}} \int \mathbb{1}_{A_n}(z + y) e^{-\frac{1}{2(t_n - t_{n-1})} z^2}\, dz \\
&= \mathbb{E}\, \mathbb{1}_{A_n}(B(t_n) - B(t_{n-1}) + y)
\end{aligned}
$$

In the next iteration we get

$$
\begin{aligned}
&\mathbb{E}\, \mathbb{1}_{A_n}\Big((B(t_n) - B(t_{n-1})) + (B(t_{n-1}) - B(t_{n-2}) + y)\Big) \mathbb{1}_{A_{n-1}}\Big((B(t_{n-1}) - B(t_{n-2}) + y)\Big) \\
&= \mathbb{E}\, \mathbb{1}_{A_n}\Big((B(t_n) - B(t_{n-2}) + y)\Big) \mathbb{1}_{A_{n-1}}\Big((B(t_{n-1}) - B(t_{n-2}) + y)\Big)
\end{aligned}
$$

etc. and we finally arrive at

$$\mathbb{Q}(W(t_j) \in A_j, \forall j = 1, \dots, n) = \mathbb{E} \prod_{j=1}^{n} \mathbb{1}_{A_j}\Big(\textstyle\sum_{k=1}^{j} (B(t_k) - B(t_{k-1})) \Big).$$

Solution 2: As in the first part of Solution 1 we see that we can assume that $T = t_n$. Since we know the joint distribution of $(B(t_1), \dots, B(t_n))$, cf. (2.10b), we get (using $x_0 = t_0 = 0$)

$$\begin{aligned}
&\mathbb{Q}\big(W(t_1) \in A_1, \dots, W(t_n) \in A_n\big)\\
&= \int \prod_{j=1}^{n} \mathbb{1}_{A_j}(B(t_j) - \xi t_j)\, e^{\xi B(t_n) - \frac{1}{2}\xi^2 t_n}\, d\mathbb{P}\\
&= \int \cdots \int \prod_{j=1}^{n} \mathbb{1}_{A_j}(x_j - \xi t_j)\, e^{\xi x_n - \frac{1}{2}\xi^2 t_n}\, e^{-\frac{1}{2}\sum_{j=1}^{n} \frac{(x_j - x_{j-1})^2}{t_j - t_{j-1}}} \frac{dx_1 \dots dx_n}{(2\pi)^{n/2} \prod_{j=1}^{n} \sqrt{t_j - t_{j-1}}}\\
&= \int \cdots \int \left(\prod_{j=1}^{n} \left[\mathbb{1}_{A_j}(x_j - \xi t_j)\, e^{-\frac{1}{2} \frac{(x_j - x_{j-1})^2}{t_j - t_{j-1}}} \right] \right) e^{\sum_{j=1}^{n} \left(\xi(x_j - x_{j-1}) - \frac{1}{2}\xi^2 (t_j - t_{j-1})\right)} \frac{dx_1 \dots dx_n}{(2\pi)^{n/2} \prod_{j=1}^{n} \sqrt{t_j - t_{j-1}}}\\
&= \int \cdots \int \prod_{j=1}^{n} \left[\mathbb{1}_{A_j}(x_j - \xi t_j)\, e^{-\frac{1}{2} \frac{(x_j - x_{j-1})^2}{t_j - t_{j-1}} + \xi(x_j - x_{j-1}) - \frac{1}{2}\xi^2 (t_j - t_{j-1})} \right] \frac{dx_1 \dots dx_n}{(2\pi)^{n/2} \prod_{j=1}^{n} \sqrt{t_j - t_{j-1}}}\\
&= \int \cdots \int \prod_{j=1}^{n} \left[\mathbb{1}_{A_j}(x_j - \xi t_j)\, e^{-\frac{1}{2(t_j - t_{j-1})} \left((x_j - x_{j-1}) + \xi(t_j - t_{j-1})\right)^2} \right] \frac{dx_1 \dots dx_n}{(2\pi)^{n/2} \prod_{j=1}^{n} \sqrt{t_j - t_{j-1}}}\\
&= \int \cdots \int \prod_{j=1}^{n} \left[\mathbb{1}_{A_j}(z_j)\, e^{-\frac{1}{2(t_j - t_{j-1})} (z_j - z_{j-1})^2} \right] \frac{dz_1 \dots dz_n}{(2\pi)^{n/2} \prod_{j=1}^{n} \sqrt{t_j - t_{j-1}}}\\
&= \mathbb{P}\left(B(t_1) \in A_1, \dots, B(t_n) \in A_n\right).
\end{aligned}$$

Problem 17.4 (Solution) We have

$$\begin{aligned}
&\mathbb{P}\Big(B_t + \alpha t \leqslant x,\ \sup_{s \leqslant t}(B_s + \alpha s) \leqslant y\Big)\\
&= \int \mathbb{1}_{(-\infty,x]}\big(B_t + \alpha t\big) \mathbb{1}_{(-\infty,y]}\big(\sup_{s\leqslant t}(B_s + \alpha s)\big)\, d\mathbb{P}\\
&= \int \mathbb{1}_{(-\infty,x]}\big(B_t + \alpha t\big) \mathbb{1}_{(-\infty,y]}\big(\sup_{s\leqslant t}(B_s + \alpha s)\big)\, \frac{1}{\beta_t}\, d\mathbb{Q}
\end{aligned}$$

where $\mathbb{Q} = \beta_t \cdot \mathbb{P}$ with $\beta_t = \exp\left(-\alpha B_t - \frac{1}{2}\alpha^2 t\right)$

$$\begin{aligned}
&= \int \mathbb{1}_{(-\infty,x]}\big(B_t + \alpha t\big) \mathbb{1}_{(-\infty,y]}\big(\sup_{s\leqslant t}(B_s + \alpha s)\big)\, e^{\alpha B_t + \frac{1}{2}\alpha^2 t}\, d\mathbb{Q}\\
&= \int \mathbb{1}_{(-\infty,x]}\big(B_t + \alpha t\big) \mathbb{1}_{(-\infty,y]}\big(\sup_{s\leqslant t}(B_s + \alpha s)\big)\, e^{\alpha(B_t + \alpha t)}\, e^{-\frac{1}{2}\alpha^2 t}\, d\mathbb{Q}\\
&\overset{\text{Girsanov}}{=} e^{-\frac{1}{2}\alpha^2 t} \int \mathbb{1}_{(-\infty,x]}\big(W_t\big) \mathbb{1}_{(-\infty,y]}\big(\sup_{s\leqslant t} W_s\big)\, e^{\alpha W_t}\, d\mathbb{Q}\\
&= e^{-\frac{1}{2}\alpha^2 t} \int_{\mathbb{R}^d} \mathbb{1}_{(-\infty,x]}(\xi)\, e^{\alpha\xi}\, \mathbb{Q}\big(W_t \in d\xi,\ \sup_{s\leqslant t} W_s \leqslant y\big).
\end{aligned}$$

where $(W_s)_{s\leqslant t}$ is a Brownian motion for the probability measure $\mathbb{Q}$.

From Solution 2 of Problem 6.8 (or with Theorem 6.18) we have

$$\mathbb{Q}\big(\sup_{s\leqslant t} W_t < y, W_t \in d\xi\big) = \lim_{a\to-\infty} \mathbb{Q}\big(\inf_{s\leqslant t} W_s > a, \sup_{s\leqslant t} W_t < y, W_t \in d\xi\big)$$

$$\overset{(6.19)}{=} \frac{d\xi}{\sqrt{2\pi t}}\left[e^{-\frac{\xi^2}{2t}} - e^{-\frac{(\xi-2y)^2}{2t}}\right]$$

and we get the same result for $\mathbb{Q}\big(\sup_{s\leqslant t} W_t \leqslant y, W_t \in d\xi\big)$. Thus,

$$\begin{aligned}
&\mathbb{P}\Big(B_t + \alpha t \leqslant x,\ \sup_{s\leqslant t}(B_s + \alpha s) \leqslant y\Big)\\
&= \int_{-\infty}^{x} e^{\alpha\xi} e^{-\frac{1}{2}t\alpha^2} \frac{1}{\sqrt{2\pi t}}\left(e^{-\frac{\xi^2}{2t}} - e^{-\frac{(\xi-2y)^2}{2t}}\right) d\xi\\
&= \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{x}\left(e^{-\frac{(\xi-\alpha t)^2}{2t}} - e^{2\alpha y}\, e^{-\frac{(\xi-2y-\alpha t)^2}{2t}}\right) d\xi\\
&= \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\frac{x-\alpha t}{\sqrt{t}}} e^{-\frac{z^2}{2}}\,dz - \frac{e^{2\alpha y}}{\sqrt{2\pi t}}\int_{-\infty}^{\frac{x-2y-\alpha t}{\sqrt{t}}} e^{-\frac{z^2}{2}}\,dz\\
&= \Phi\big(\tfrac{x-\alpha t}{\sqrt{t}}\big) - e^{2\alpha y}\,\Phi\big(\tfrac{x-2y-\alpha t}{\sqrt{t}}\big).
\end{aligned}$$

Problem 17.5 (Solution) a) Since X_t has continuous sample paths we find that

$$\widehat{\tau}_b = \inf\big\{t \geqslant 0 : X_t \geqslant b\big\}.$$

Moreover, we have

$$\big\{\widehat{\tau}_b \leqslant t\big\} = \big\{\sup_{s\leqslant t} X_s \geqslant b\big\}.$$

Indeed,

$$\begin{aligned}
\omega \in \big\{\sup_{s\leqslant t} X_s \geqslant b\big\} &\Longrightarrow \exists s \leqslant t : X_s(\omega) \geqslant b \qquad \text{(continuous paths!)}\\
&\Longrightarrow \widehat{\tau}_b(\omega) \leqslant t\\
&\Longrightarrow \omega \in \big\{\widehat{\tau}_b \leqslant t\big\},
\end{aligned}$$

and so $\big\{\widehat{\tau}_b \leqslant t\big\} \supset \big\{\sup_{s\leqslant t} X_s \geqslant b\big\}$. Conversely,

$$\begin{aligned}
\omega \in \big\{\widehat{\tau}_b \leqslant t\big\} &\Longrightarrow \widehat{\tau}_b(\omega) \leqslant t\\
&\Longrightarrow X_{\widehat{\tau}_b(\omega)}(\omega) \geqslant b,\ \widehat{\tau}_b(\omega) \leqslant t\\
&\Longrightarrow \sup_{s\leqslant t} X_s(\omega) \geqslant b\\
&\Longrightarrow \omega \in \big\{\sup_{s\leqslant t} X_s \geqslant b\big\},
\end{aligned}$$

and so $\big\{\widehat{\tau}_b \leqslant t\big\} \subset \big\{\sup_{s\leqslant t} X_s \geqslant b\big\}$.

By the previous problem, Problem 17.4, $\mathbb{P}(\sup_{s\leqslant t} X_s = b) = 0$. This means that

$$\begin{aligned}
\mathbb{P}\left(\widehat{\tau}_b > t\right) &= \mathbb{P}\left(\sup_{s\leqslant t} X_s < b\right)\\
&= \mathbb{P}\left(\sup_{s\leqslant t} X_s \leqslant b\right)\\
&= \mathbb{P}\left(X_t \leqslant b,\ \sup_{s\leqslant t} X_s \leqslant b\right)\\
&\underset{17.4}{\overset{\text{Prob.}}{=}} \Phi\Big(\tfrac{b-\alpha t}{\sqrt{t}}\Big) - e^{2\alpha b}\Phi\Big(\tfrac{-b-\alpha t}{\sqrt{t}}\Big)\\
&= \Phi\Big(\tfrac{b}{\sqrt{t}} - \alpha\sqrt{t}\Big) - e^{2\alpha b}\Phi\Big(-\tfrac{b}{\sqrt{t}} - \alpha\sqrt{t}\Big).
\end{aligned}$$

Differentiating in t yields

$$\begin{aligned}
-\frac{d}{dt}\,\mathbb{P}\left(\widehat{\tau}_b > t\right) &= e^{2\alpha b}\Big(\tfrac{b}{2t\sqrt{t}} - \tfrac{\alpha}{2\sqrt{t}}\Big)\,\Phi'\Big(-\tfrac{b}{\sqrt{t}} - \alpha\sqrt{t}\Big) + \Big(\tfrac{b}{2t\sqrt{t}} + \tfrac{\alpha}{2\sqrt{t}}\Big)\,\Phi'\Big(\tfrac{b}{\sqrt{t}} - \alpha\sqrt{t}\Big)\\
&= \frac{1}{\sqrt{2\pi}}\left(e^{2\alpha b}\Big(\tfrac{b}{2t\sqrt{t}} - \tfrac{\alpha}{2\sqrt{t}}\Big)e^{-\frac{(b+\alpha t)^2}{2t}} + \Big(\tfrac{b}{2t\sqrt{t}} + \tfrac{\alpha}{2\sqrt{t}}\Big)e^{-\frac{(b-\alpha t)^2}{2t}}\right)\\
&= \frac{1}{\sqrt{2\pi}}\left(\Big(\tfrac{b}{2t\sqrt{t}} - \tfrac{\alpha}{2\sqrt{t}}\Big)e^{-\frac{(b-\alpha t)^2}{2t}} + \Big(\tfrac{b}{2t\sqrt{t}} + \tfrac{\alpha}{2\sqrt{t}}\Big)e^{-\frac{(b-\alpha t)^2}{2t}}\right)\\
&= \frac{1}{\sqrt{2\pi}}\,\frac{2b}{2t\sqrt{t}}\,e^{-\frac{(b-\alpha t)^2}{2t}}\\
&= \frac{b}{t\sqrt{2\pi t}}\,e^{-\frac{(b-\alpha t)^2}{2t}}
\end{aligned}$$

b) We have seen in part a) that

$$\mathbb{P}\left(\widehat{\tau}_b > t\right) = \Phi\Big(\tfrac{b-\alpha t}{\sqrt{t}}\Big) - e^{2\alpha b}\,\Phi\Big(\tfrac{-b-\alpha t}{\sqrt{t}}\Big)$$

$$\xrightarrow[t\to\infty]{} \begin{cases} \Phi(-\infty) - e^{2\alpha b}\Phi(-\infty) = 0 & \text{if } \alpha > 0\\ \Phi(0) - e^{0}\Phi(0) = 0 & \text{if } \alpha = 0\\ \Phi(\infty) - e^{2\alpha b}\Phi(\infty) = 1 - e^{2\alpha b} & \text{if } \alpha < 0 \end{cases}$$

Therefore, we get

$$\mathbb{P}\left(\widehat{\tau}_b < \infty\right) = \begin{cases} 1 & \text{if } \alpha \geqslant 0\\ e^{2\alpha b} & \text{if } \alpha < 0. \end{cases}$$

Problem 17.6 (Solution) Basically, this is done on page 260, first few lines. If you want to be a bit more careful, you should treat the real and imaginary parts of $\exp[i\xi B_T] = \cos(\xi B_T) + i\sin(\xi B_t)$ separately. Let us do this for the real part.

We apply the 2-dimensional Itô-formula to the process $X_t = (t, B_t)$ and with $f(t,x) = \cos(\xi x)e^{t\xi^2/2}$ (see also Problem 16.3): Since

$$\begin{aligned}
\partial_t f(t,x) &= \frac{\xi^2}{2}\cos(\xi x)e^{t\xi^2/2}\\
\partial_x f(t,x) &= -\xi\sin(\xi x)e^{t\xi^2/2}\\
\partial_x^2 f(t,x) &= -\xi^2\cos(\xi x)e^{t\xi^2/2}
\end{aligned}$$

we get

$$\begin{aligned}
&\cos(\xi B_T)e^{T\xi^2/2} - 1\\
&= \frac{\xi^2}{2}\int_0^T \cos(\xi B_s)e^{s\xi^2/2}\,ds - \xi\int_0^T \sin(\xi B_s)e^{s\xi^2/2}\,dB_s - \frac{1}{2}\xi^2\int_0^T \cos(\xi B_s)e^{s\xi^2/2}\,ds\\
&= -\xi\int_0^T \sin(\xi B_s)e^{s\xi^2/2}\,dB_s.
\end{aligned}$$

Thus,

$$\cos(\xi B_T) = e^{-T\xi^2/2} - \xi\int_0^T \sin(\xi B_s)e^{(s-T)\xi^2/2}\,dB_s.$$

Since the integrand of the stochastic integral is continuous and bounded, it is clear that it is in $L^2_{\mathcal{P}}(\lambda_T \otimes \mathbb{P})$. Hence $\cos(\xi B_t) \in \mathcal{H}^2_T$.

The imaginary part can be treated in a similar way.

Problem 17.7 (Solution) Because of the properties of conditional expectations we have for $s \leqslant t$

$$\mathbb{E}\left(M_t \,\middle|\, \mathcal{H}_s\right) = \mathbb{E}\left(M_t \,\middle|\, \sigma(\mathcal{F}_s, \mathcal{G}_s)\right) \overset{M \perp\!\!\!\perp \mathcal{G}_\infty}{=} \mathbb{E}\left(M_t \,\middle|\, \mathcal{F}_s\right) = M_s.$$

Thus, $(M_t, \mathcal{H}_t)_{t\geqslant 0}$ is still a martingale; $(B_t, \mathcal{H}_t)_{t\geqslant 0}$ is treated in a similar way.

Problem 17.8 (Solution) Recall that

$$\tau(s) = \inf\{t \geqslant 0 : a(t) > s\}.$$

Since for any $\epsilon > 0$

$$\{t : a(t) \geqslant s\} \subset \{t : a(t) > s - \epsilon\} \subset \{t : a(t) \geqslant s - \epsilon\}$$

we get

$$\inf\{t : a(t) \geqslant s\} \geqslant \inf\{t : a(t) > s - \epsilon\} \geqslant \inf\{t : a(t) \geqslant s - \epsilon\}$$

and

$$\inf\{t : a(t) \geqslant s\} \geqslant \underbrace{\lim_{\epsilon \uparrow 0} \inf\{t : a(t) > s - \epsilon\}}_{= \lim_{\epsilon \uparrow 0} \tau(s-\epsilon) = \tau(s-)} \geqslant \lim_{\epsilon \uparrow 0} \inf\{t : a(t) \geqslant s - \epsilon\}.$$

Thus, $\inf\{t : a(t) \geqslant s\} \geqslant \tau(s-)$. Assume that $\inf\{t : a(t) \geqslant s\} > \tau(s-)$. Then

$$a(\tau(s-)) < s.$$

On the other hand, by Lemma 17.14 b)

$$s - \epsilon \leqslant a(\tau(s-\epsilon)) \leqslant a(\tau(s-)) < s \qquad \forall \epsilon > 0.$$

This leads to a contradiction, and so $\inf\{t : a(t) \geqslant s\} \leqslant \tau(s-)$.

The proof for $a(s-)$ is similar.

Assume that $\tau(s) \geqslant t$. Then $a(t-) = \inf\{s \geqslant 0 : \tau(s) \geqslant t\} \leqslant s$. On the other hand,

$$a(t-) \leqslant s \implies \forall \epsilon > 0 : a(t-\epsilon) \leqslant s \overset{17.14\text{ d)}}{\implies} \forall \epsilon > 0 : \tau(s) > t - \epsilon \implies \tau(s) \geqslant t.$$

Problem 17.9 (Solution) We have

$$\begin{aligned} \{\langle M\rangle_t \leqslant s\} = \bigcap_{n\geqslant 1} \{\langle M\rangle_t < s + 1/n\} &= \bigcap_{n\geqslant 1} \{\langle M\rangle_t \geqslant s + 1/n\}^c \\ &\underset{\text{c)}}{\overset{17.14}{=}} \bigcap_{n\geqslant 1} \{\tau_{s+1/n-} \leqslant s\}^c \in \bigcap_{n\geqslant 1} \mathcal{F}_{\tau_{s+1/n}} \overset{A.15}{=} \mathcal{F}_{\tau_s+}. \end{aligned}$$

As $\mathcal{F}_t$ is right-continuous, $\mathcal{F}_{\tau_s+} = \mathcal{F}_{\tau_s} = \mathcal{G}_s$ and we conclude that $\langle M\rangle_t$ is a $\mathcal{G}_t$ stopping time.

Problem 17.10 (Solution) <u>*Solution 1:*</u> Assume that $f \in \mathcal{C}^2$. Then we can apply Itô's formula. Use Itô's formula for the deterministic process $X_t = f(t)$ and apply it to the function x^a (we assume that $f \geqslant 0$ to make sure that f^a is defined for all $a > 0$):

$$f^a(t) - f^a(0) = \int_0^t \left[\frac{d}{dx}x^a\right]_{x=f(s)} df(s) = \int_0^t a f^{a-1}(s)\,df(s).$$

This proves that the primitive $\int f^{a-1}\,df = f^a/a$. The rest is an approximation argument ($f \in \mathcal{C}^1$ is pretty immediate).

<u>*Solution 2:*</u> Any absolutely continuous function has an Lebesgue a.e. defined derivative f' and $f = \int f'\,ds$. Thus,

$$\int_0^t f^{a-1}(s)\,df(s) = \int_0^t f^{a-1}(s)f'(s)\,ds = \int_0^t \frac{1}{a}\frac{d}{ds}f^a(s)\,ds = \left[\frac{f^a(s)}{a}\right]_0^t = \frac{f^a(t) - f^a(0)}{a}.$$

Problem 17.11 (Solution) **Theorem.** *Let $B_t = (B_t^1, \dots, B_t^d)$ be a d-dimensional Brownian motion and $f_1, \dots, f_d \in L^2_{\mathcal{P}}(\lambda_T \otimes \mathbb{P})$ for all $T > 0$. Then, we have for $2 \leqslant p < \infty$*

$$\mathbb{E}\left[\left(\int_0^T \sum_{k=1}^d |f_k(s)|^2\,ds\right)^{p/2}\right] \asymp \mathbb{E}\left[\sup_{t\leqslant T}\left|\sum_k \int_0^t f_k(s)\,dB_s^k\right|^p\right] \tag{17.2}$$

with finite comparison constants which depend only on p.

Proof. Let $X_t = \sum_k \int_0^t f_k(s)\,dB_s^k$. Then we have

$$\begin{aligned}
\langle X\rangle_t &= \left\langle \sum_k \int_0^t f_k(s)\,dB_s^k,\ \sum_l \int_0^t f_l(s)\,dB_s^l \right\rangle \\
&= \sum_{k,l} \left\langle \int_0^t f_k(s)\,dB_s^k,\ \int_0^t f_l(s)\,dB_s^l \right\rangle \\
&= \sum_{k,l} \int_0^t f_k(s)f_l(s)\,d\left\langle B^k,\ B^l\right\rangle_s \\
&= \sum_k \int_0^t f_k^2(s)\,ds
\end{aligned}$$

since $dB_s^k dB_s^l = d\langle B^k, B^l\rangle_s = \delta_{kl}\,ds$.

With these notations, the proof of Theorem 17.16 goes through almost unchanged and we get the inequalities for $p \geqslant 2$. □

Remark: Often one needs only one direction (as we do later in the book) and one can use 17.18 directly, without going through the proof again. Note that

$$\begin{aligned}
\left|\sum_{k=1}^d \int_0^t f_k(s)\,dB_s^k\right|^p &\leqslant \left(\sum_{k=1}^d \left|\int_0^t f_k(s)\,dB_s^k\right|\right)^p \\
&\leqslant c_{d,p}\sum_{k=1}^d \left|\int_0^t f_k(s)\,dB_s^k\right|^p.
\end{aligned}$$

Thus, by (17.18)

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\leqslant T}\left|\sum_{k=1}^{d}\int_0^t f_k(s)\,dB_s^k\right|^p\right] &\leqslant c_{d,p}\sum_{k=1}^{d}\mathbb{E}\left[\sup_{t\leqslant T}\left|\int_0^t f_k(s)\,dB_s^k\right|^p\right]\\
&\asymp c_{d,p}\sum_{k=1}^{d}\mathbb{E}\left[\left(\int_0^T |f_k(s)|^2\,ds\right)^{p/2}\right]\\
&\asymp c_{d,p}\,\mathbb{E}\left[\left(\int_0^T \sum_{k=1}^{d}|f_k(s)|^2\,ds\right)^{p/2}\right].
\end{aligned}$$

18 Stochastic Differential Equations

Problem 18.1 (Solution) We have

$$dX_t = b(t)\,dt + \sigma(t)\,dB_t$$

where b, σ are non-random coefficients such that the corresponding (stochastic) integrals exist. Obviously,

$$(dX_t)^2 = \sigma^2(t)\,(dB_t)^2 = \sigma^2(t)\,dt$$

and we get for $0 \leqslant s \leqslant t < \infty$, using Itô's formula,

$$\begin{aligned}
e^{i\xi X_t} - e^{i\xi X_s} &= \int_s^t i\xi e^{i\xi X_r}\,b(r)\,dr + \int_s^t i\xi e^{i\xi X_r}\,\sigma(r)\,dB_r \\
&\quad - \frac{1}{2}\int_s^t \xi^2 e^{i\xi X_r}\,\sigma^2(r)\,dr.
\end{aligned}$$

Now take any $F \in \mathcal{F}_s$ and multiply both sides of the above formula by $e^{-\xi X_s}\mathbb{1}_F$. We get

$$\begin{aligned}
e^{i\xi(X_t - X_s)}\mathbb{1}_F - \mathbb{1}_F &= \int_s^t i\xi e^{i\xi(X_r - X_s)}\mathbb{1}_F\,b(r)\,dr + \int_s^t i\xi e^{i\xi(X_r - X_s)}\mathbb{1}_F\,\sigma(r)\,dB_r \\
&\quad - \frac{1}{2}\int_s^t \xi^2 e^{i\xi(X_r - X_s)}\mathbb{1}_F\,\sigma^2(r)\,dr.
\end{aligned}$$

Taking expectations gives

$$\begin{aligned}
\mathbb{E}\left(e^{i\xi(X_t - X_s)}\mathbb{1}_F\right) &= \mathbb{P}(F) + \int_s^t i\xi\,\mathbb{E}\left(e^{i\xi(X_r - X_s)}\mathbb{1}_F\right) b(r)\,dr \\
&\quad - \frac{1}{2}\int_s^t \xi^2\,\mathbb{E}\left(e^{i\xi(X_r - X_s)}\mathbb{1}_F\right)\sigma^2(r)\,dr \\
&= \mathbb{P}(F) + \int_s^t \left(i\xi b(r) - \frac{1}{2}\,\xi^2\,\sigma^2(r)\right)\mathbb{E}\left(e^{i\xi(X_r - X_s)}\mathbb{1}_F\right) dr.
\end{aligned}$$

Define $\phi_{s,t}(\xi) := \mathbb{E}\left(e^{i\xi(X_t - X_s)}\mathbb{1}_F\right)$. Then the integral equation

$$\phi_{s,t}(\xi) = \mathbb{P}(F) + \int_s^t \left(i\xi b(r) - \frac{1}{2}\,\xi^2\,\sigma^2(r)\right)\phi_{r,s}(\xi)\,dr$$

has the unique solution (use Gronwall's lemma, cf. also the proof of Theorem 17.5)

$$\phi_{s,t}(\xi) = \mathbb{P}(F)\,e^{i\xi\int_s^t b(s)\,ds - \frac{1}{2}\xi^2\int_s^t \sigma^2(r)\,dr}$$

and so

$$\mathbb{E}\left(e^{i\xi(X_t - X_s)}\mathbb{1}_F\right) = \mathbb{P}(F)\,e^{i\xi\int_s^t b(r)\,dr - \frac{1}{2}\xi^2\int_s^t \sigma^2(r)\,dr}. \tag{*}$$

If we take in (*) $F = \Omega$ and $s = 0$, we see that

$$X_t \sim \mathsf{N}(\mu_t, \sigma_t^2), \qquad \mu_t = \int_0^t b(r)\,dr, \qquad \sigma_t^2 = \frac{1}{2}\int_0^t \sigma^2(r)\,dr.$$

If we take in (*) $F = \Omega$ then the increment satisfies $X_t - X_s \sim \mathsf{N}(\mu_t - \mu_s, \sigma_t^2 - \sigma_s^2)$. If F is arbitrary, (*) shows that

$$X_t - X_s \perp\!\!\!\perp \mathcal{F}_s,$$

see the Lemma at the end of this section.

The above considerations show that

$$\mathbb{E}\, e^{\sum_{j=1}^n \xi_j (X_{t_j} - X_{t_{j-1}})} = \prod_{j=1}^n \exp\left(i\xi \int_{t_{j-1}}^{t_j} b(r)\,dr - \frac{1}{2}\xi^2 \int_{t_{j-1}}^{t_j} \sigma^2(r)\,dr \right),$$

i.e. $(X_{t_1}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}})$ is a Gaussian random vector with independent components. Since $X_{t_k} = \sum_{j=1}^k (X_{t_j} - X_{t_{j-1}})$ we see that $(X_{t_1}, \dots, X_{t_n})$ is a Gaussian random variable.

Let us, finally, compute $\mathbb{E}(X_s X_t)$. By independence, we have

$$\begin{aligned}
\mathbb{E}(X_s X_t) &= \mathbb{E}(X_s^2) + \mathbb{E}\, X_s (X_t - X_s) \\
&= \mathbb{E}(X_s^2) + \mathbb{E}\, X_s\, \mathbb{E}(X_t - X_s) \\
&= \mathbb{E}(X_s^2) + \mathbb{E}\, X_s\, \mathbb{E}\, X_t - (\mathbb{E}\, X_s)^2 \\
&= \mathbb{V}\, X_s + \mathbb{E}\, X_s\, \mathbb{E}\, X_t \\
&= \int_0^s \sigma^2(r)\,dr + \int_0^s b(r)\,dr \int_0^t b(r)\,dr.
\end{aligned}$$

In fact, since the mean is not zero, it would have been more elegant to compute the covariance

$$\operatorname{Cov}(X_s, X_t) = \mathbb{E}(X_s - \mu_s)(X_t - \mu_t) = \mathbb{E}(X_s X_t) - \mathbb{E}\, X_s\, \mathbb{E}\, X_t = \mathbb{V}\, X_s = \int_0^s \sigma^2(r)\,dr.$$

Lemma. *Let X be a random variable and $\mathcal{F}$ a σ field. Then*

$$\mathbb{E}\left(e^{i\xi X} \mathbb{1}_F\right) = \mathbb{E}\, e^{i\xi X} \cdot \mathbb{P}(F) \quad \forall \xi \in \mathbb{R} \implies X \perp\!\!\!\perp \mathcal{F}.$$

Proof. Note that $e^{i\eta \mathbb{1}_F} = e^{i\eta} \mathbb{1}_F + \mathbb{1}_{F^c}$. Thus,

$$\begin{aligned}
\mathbb{E}\left(e^{i\xi X}\, \mathbb{1}_{F^c}\right) &= \mathbb{E}\left(e^{i\xi X}\right) - \mathbb{E}\left(e^{i\xi X}\, \mathbb{1}_F\right) \\
&= \mathbb{E}\left(e^{i\xi X}\right) - \mathbb{E}\left(e^{i\xi X}\right) \mathbb{P}(F) \\
&= \mathbb{E}\left(e^{i\xi X}\right) \mathbb{P}(F^c)
\end{aligned}$$

and this implies

$$\mathbb{E}\left(e^{i\xi X}\, e^{i\eta \mathbb{1}_F}\right) = \mathbb{E}\left(e^{i\xi X}\right) \mathbb{E}\left(e^{i\eta \mathbb{1}_F}\right) \qquad \forall \xi, \eta \in \mathbb{R}.$$

This shows that $X \perp\!\!\!\perp \mathbb{1}_F$ and $X \perp\!\!\!\perp F$ for all $F \in \mathcal{F}$. $\square$

Problem 18.2 (Solution) a) We have $\Delta t = 2^{-n}$ and

$$\Delta X_n(t_{k-1}) = X_n(t_k) - X_n(t_{k-1}) = -\tfrac{1}{2}\, X_n(t_{k-1})\, 2^{-n} + B(t_k) - B(t_{k-1})$$

and this shows

$$\begin{aligned}
X_n(t_k) &= X_n(t_{k-1}) - \tfrac{1}{2}\, X_n(t_{k-1})\, 2^{-n} + B(t_k) - B(t_{k-1})\\
&= (1-2^{-n-1})\, X_n(t_{k-1}) + B(t_k) - B(t_{k-1})\\
&= (1-2^{-n-1})\big[(1-2^{-n-1})\, X_n(t_{k-2}) + B(t_{k-1}) - B(t_{k-2})\big] + \big[B(t_k) - B(t_{k-1})\big]\\
&\vdots\\
&= (1-2^{-n-1})^k X_n(t_0) + (1-2^{-n-1})^{k-1}\big[B(t_1) - B(t_0)\big] + \ldots +\\
&\quad + (1-2^{-n-1})\big[B(t_{k-1}) - B(t_{k-2})\big] + \big[B(t_k) - B(t_{k-1})\big]\\
&= (1-2^{-n-1})^k A + \sum_{j=1}^{k-1} (1-2^{-n-1})^j \big[B(t_{k-j}) - B(t_{k-j-1})\big]
\end{aligned}$$

Observe that $B(t_j) - B(t_{j-1}) \sim \mathsf{N}(0, 2^{-n})$ for all j and $A \sim \mathsf{N}(0,1)$. Because of the independence we get

$$X_n(t_n) = X_n(k2^{-n}) \sim \mathsf{N}\Big(0,\ (1-2^{-n-1})^{2k} + \textstyle\sum_{j=1}^{k-1} (1-2^{-n-1})^{2j} \cdot 2^{-n}\Big)$$

For $k = 2^{n-1}$ we get $t_k = \frac{1}{2}$ and so

$$X_n\big(\tfrac{1}{2}\big) \sim \mathsf{N}\Big(0,\ (1-2^{-n-1})^{2^n} + \textstyle\sum_{j=1}^{2^n-1} (1-2^{-n-1})^{2j} \cdot 2^{-n}\Big).$$

Using

$$\lim_{n\to\infty} (1-2^{-n-1})^{2^n} = e^{-\frac{1}{2}}$$

and

$$\sum_{j=1}^{2^n-1} (1-2^{-n-1})^{2j} \cdot 2^{-n} = \frac{1-(1-2^{-n-1})^{2^n}}{1-(1-2^{-n-1})^2} \cdot 2^{-n} = \frac{1-(1-2^{-n-1})^{2^n}}{1-2^{-n-2}} \xrightarrow[n\to\infty]{} 1 - e^{-\frac{1}{2}}$$

finally shows that $X_n\big(\frac{1}{2}\big) \xrightarrow[n\to\infty]{d} X \sim \mathsf{N}(0,1)$.

b) The solution of this SDE follows along the lines of Example 18.4 where $\alpha(t) \equiv 0$, $\beta(t) \equiv -\frac{1}{2}$, $\delta(t) \equiv 0$ and $\gamma(t) \equiv 1$:

$$\begin{aligned}
dX_t^\circ &= \tfrac{1}{2}\, X_t^\circ\, dt \implies X_t^\circ = e^{t/2}\\
Z_t &= e^{t/2}\, X_t, \quad Z_0 = X_0\\
dZ_t &= e^{t/2}\, dB_t \implies Z_t = Z_0 + \int_0^t e^{s/2}\, dB_s\\
X_t &= e^{-t/2}\, A + e^{-t/2} \int_0^t e^{s/2}\, dB_s.
\end{aligned}$$

For $t = \frac{1}{2}$ we get

$$X_{1/2} = A\, e^{-1/4} + e^{-1/4} \int_0^{1/2} e^{s/2}\, dB_s$$

$$\implies X_{1/2} \sim \mathsf{N}\Big(0,\, e^{-1/2} + e^{-1/2}\textstyle\int_0^{1/2} e^s\,ds\Big) = \mathsf{N}(0,1).$$

So, we find for all $s \leqslant t$

$$\begin{aligned}
C(s,t) = \mathbb{E}\, X_s X_t &= e^{-s/2}e^{-t/2}\,\mathbb{E}\, A^2 + e^{-s/2}e^{-t/2}\,\mathbb{E}\left(\int_0^s e^{r/2}\,dB_r \int_0^t e^{u/2}\,dB_u\right)\\
&= e^{-(s+t)/2} + e^{-(s+t)/2}\int_0^s e^r\,dr\\
&= e^{-(t-s)/2}.
\end{aligned}$$

This finally shows that $C(s,t) = e^{-|t-s|/2}$.

Problem 18.3 (Solution) Since X_t° is such that $1/X_t^\circ$ solves the homogeneous SDE from Example 18.3, we see that

$$X_t^\circ = \exp\left(-\int_0^t \left(\beta(s) - \tfrac{1}{2}\,\delta^2(s)\right)\,ds\right)\exp\left(-\int_0^t \delta(s)\,dB_s\right)$$

(mind that the 'minus' sign comes from $1/X_t^\circ$).

Observe that $X_t^\circ = f(I_t^1, I_t^2)$ where I_t is an Itô process with

$$\begin{aligned}
I_t^1 &= -\int_0^t \left(\beta(s) - \tfrac{1}{2}\,\delta^2(s)\right)\,ds\\
I_t^2 &= -\int_0^t \delta(s)\,dB_s.
\end{aligned}$$

Now we get from Itô's multiplication table

$$dI_t^1 dI_t^1 = dI_t^1 dI_t^2 = 0 \quad\text{and}\quad dI_t^2 dI_t^2 = \delta^2(t)\,dt$$

and, by Itô's formula

$$\begin{aligned}
dX_t^\circ &= \partial_1 f(I_t^1, I_t^2)\,dI_t^1 + \partial_2 f(I_t^1, I_t^2)\,dI_t^2 + \tfrac{1}{2}\sum_{j,k=1}^{2} \partial_k\partial_k\, dI_t^j\, dI_t^k\\
&= X_t^\circ\left(dI_t^1 + dI_t^2 + \tfrac{1}{2}\,dI_t^2\,dI_t^2\right)\\
&= X_t^\circ\left(-\beta(t)\,dt + \tfrac{1}{2}\,\delta^2(t)\,dt - \delta(t)\,dB_t + \tfrac{1}{2}\,\delta^2(t)\,dt\right)\\
&= X_t^\circ(-\beta(t) + \delta^2(t))\,dt - X_t^\circ\delta(t)\,dB_t.
\end{aligned}$$

<u>*Remark:*</u>

1. we used here the two-dimensional Itô formula (16.6) but we could have equally well used the one-dimensional version (16.6) with the Itô process $I_t^1 + I_t^2$.
2. observe that Itô's multiplication table gives us exactly the second-order term in (16.6).

Since

$$dZ_t = (\alpha(t) - \gamma(t)\delta(t))X_t^\circ\,dt + \gamma(t)X_t^\circ\,dB_t \quad\text{and}\quad X_t = Z_t/X_t^\circ$$

we get

$$X_t = \frac{1}{X_t^\circ}\left(X_0 + \int_0^t (\alpha(s) - \gamma(s)\delta(s))X_s^\circ\,ds + \int_0^t \gamma(s)X_s^\circ\,dB_s\right).$$

Problem 18.4 (Solution) a) We have $X_t = e^{-\beta t}X_0 + \int_0^t \sigma e^{-\beta(t-s)}\,dB_s$. This can be shown in three ways:

Solution 1: you guess the right result and use Itô's formula (16.5) to verify that the above X_t is indeed a solution to the SDE. For this rewrite the above solution as

$$e^{\beta t}X_t = X_0 + \int_0^t \sigma e^{\beta s}\,dB_s \implies d(e^{\beta t}X_t) = \sigma e^{\beta t}\,dB_t.$$

Now with the two-dimensional Itô formula for $f(x,y) = xy$ and the two-dimensional Itô-process $(e^{\beta t}, X_t)$ we get

$$d(e^{\beta t}X_t) = \beta X_t e^{\beta t}\,dt + e^{\beta t}\,dX_t$$

so that

$$\beta X_t\,e^{\beta t}\,dt + e^{\beta t}\,dX_t = \sigma e^{\beta t}\,dB_t \iff dX_t = -\beta X_t\,dt + \sigma\,dB_t.$$

Admittedly, this is unfair as one has to know the solution beforehand. On the other hand, this is exactly the way one *verifies* that the solution one has found is the correct one.

Solution 2: you apply the time-dependent Itô formula from Problem 16.3 or the 2-dimensional Itô formula, Theorem 16.6 to

$$X_t = u(t, I_t) \quad \text{and} \quad I_t = \int_0^t e^{\beta s}\,dB_s \quad \text{and} \quad u(t,x) = e^{\beta t}X_0 + \sigma e^{\beta t}x$$

to get—as $dt\,dB_t = 0$—

$$dX_t = \partial_t u(t, I_t)\,dt + \partial_x u(t, I_t)\,dI_t + \tfrac{1}{2}\,\partial_x^2\,u(t, B_t)\,dt.$$

Again, this is best for the verification of the solution since you need to know its form beforehand.

Solution 3: you use Example 18.4 with $\alpha(t) \equiv 0$, $\beta(t) \equiv -\beta$, $\gamma(t) \equiv \sigma$ and $\delta(t) \equiv 0$. But, honestly, you will have to look up the formula in the book. We get

$$\begin{aligned}
dX_t^\circ &= \beta X_t^\circ\,dt, \quad X_0^\circ = 1 \implies X_t^\circ = e^{\beta t};\\
Z_t &= e^{\beta t}\,X_t, \quad Z_0 = X_0 = \xi = \text{const.};\\
dZ_t &= \sigma e^{\beta t}\,dB_t;\\
Z_t &= \sigma \int_0^t e^{\beta s}\,dB_s + Z_0;\\
X_t &= e^{-\beta t}\,\xi + e^{-\beta t}\,\sigma \int_0^t e^{\beta s}\,dB_s, \quad t \geqslant 0.
\end{aligned}$$

Solution 4: by bare hands and with Itô's formula! Consider first the *deterministic* ODE

$$x_t = x_0 - \beta \int_0^t x_s\,ds$$

which has the solution $x_t = x_0\, e^{-\beta t}$, i.e. $e^{\beta t} x_t = x_0 = \text{const}$. This indicates that the transformation

$$Y_t := e^{\beta t} X_t$$

might be sensible. Thus, $Y_t = f(t, X_t)$ where $f(t,x) = e^{\beta t} x$. Thus,

$$\partial_t f(t,x) = \beta f(t,x) = \beta x e^{\beta t}, \quad \partial_x f(t,x) = e^{\beta t}, \quad \partial_x^2 f_{xx}(t,x) = 0.$$

By assumption,

$$dX_t = -\beta X_t\, dt + \sigma\, dB_t \implies (dX_t)^2 = \sigma^2\, (dB_t)^2 = \sigma^2\, dt,$$

and by Itô's formula (16.6) we get

$$\begin{aligned}
&Y_t - Y_0 \\
&= \int_0^t \Big(\underbrace{f_t(s,X_s) - \beta X_s f_x(s,X_s)}_{=0} + \underbrace{\tfrac{1}{2}\sigma^2 f_{xx}(s,X_s)}_{=0} \Big)\, ds + \int_0^t \sigma f_x(s,X_s)\, dB_s \\
&= \int_0^t \sigma f_x(s,X_s)\, dB_s.
\end{aligned}$$

So we have the solution, but we still have to go through the procedure in Solution 1 or 2 in order to verify our result.

b) Since X_t is the limit of normally distributed random variables, it is itself Gaussian (see also part d))—if ξ is non-random or itself Gaussian and independent of everything else. In particular, if $X_0 = \xi = \text{const.}$,

$$X_t \sim \mathsf{N}\left(e^{-\beta t}\xi,\ \sigma^2 e^{-2\beta t} \textstyle\int_0^t e^{2\beta s}\, ds\right) = \mathsf{N}\left(e^{-\beta t}\xi,\ \tfrac{\sigma^2}{2\beta}(1 - e^{-2\beta t})\right).$$

Now

$$C(s,t) = \mathbb{E}\, X_s X_t = e^{-\beta(t+s)}\, \xi^2 + \frac{\sigma^2}{2\beta}\, e^{-\beta(t+s)} (e^{2\beta s} - 1), \quad t \geqslant s \geqslant 0,$$

and, therefore

$$C(s,t) = e^{-\beta(t+s)}\, \xi^2 + \frac{\sigma^2}{2\beta} \left(e^{-\beta|t-s|} - e^{-\beta(t+s)}\right) \quad \text{for all} \ \ s,t \geqslant 0.$$

c) The asymptotic distribution, as $t \to \infty$, is $X_\infty \sim \mathsf{N}\big(0, \sigma^2 (2\beta)^{-1}\big)$.

d) We have

$$\begin{aligned}
&\mathbb{E}\left(\exp\left[i \sum_{j=1}^n \lambda_j X_{t_j}\right]\right) \\
&\quad = \mathbb{E}\left(\exp\left[i \sum_{j=1}^n \lambda_j e^{-\beta t_j} \xi + i\sigma \sum_{j=1}^n \lambda_j e^{-\beta t_j} \int_0^{t_j} e^{\beta s}\, dB_s\right]\right) \\
&\quad = \exp\left(-\frac{\sigma^2}{4\beta}\left[\sum_{j=1}^n \lambda_j e^{-\beta t_j}\right]^2\right) \mathbb{E}\left(\exp\left[i\sigma \sum_{j=1}^n \eta_j Y_j\right]\right)
\end{aligned}$$

where

$$\eta_j = \lambda_j e^{-\beta t_j}, \qquad Y_j = \int_0^{t_j} e^{\beta s}\, dB_s, \qquad t_0 = 0, \qquad Y_0 = 0.$$

Moreover,

$$\sum_{j=1}^n \eta_j Y_j = \sum_{k=1}^n (Y_k - Y_{k-1}) \sum_{j=k}^n \eta_j$$

and

$$Y_k - Y_{k-1} = \int_{t_{k-1}}^{t_k} e^{\beta s}\, dB_s \sim \mathsf{N}\big(0,\ (2\beta)^{-1}(e^{2\beta t_k} - e^{2\beta t_{k-1}})\big) \quad \text{are independent.}$$

Consequently, we see that

$$\begin{aligned}
&\mathbb{E}\left(\exp\left[i\sum_{j=1}^n \lambda_j X_{t_j}\right]\right)\\
&= \exp\left[-\frac{\sigma^2}{4\beta}\Big(\textstyle\sum_{j=1}^n \lambda_j e^{-\beta t_j}\Big)^2\right]\prod_{k=1}^n \exp\left[-\frac{\sigma^2}{4\beta}\left(e^{2\beta t_k} - e^{2\beta t_{k-1}}\right)\Big(\textstyle\sum_{j=k}^n \lambda_j e^{-\beta t_j}\Big)^2\right]\\
&= \exp\left[-\frac{\sigma^2}{4\beta}\Big(\textstyle\sum_{j=1}^n \lambda_j e^{-\beta t_j}\Big)^2\big\{1 + e^{2\beta t_1} - 1\big\}\right]\times\\
&\quad \times \prod_{k=2}^n \exp\left[-\frac{\sigma^2}{4\beta}\left(1 - e^{-2\beta(t_k - t_{k-1})}\right)\cdot\Big(\textstyle\sum_{j=k}^n \lambda_j e^{-\beta(t_j - t_k)}\Big)^2\right]\\
&= \exp\left[-\frac{\sigma^2}{4\beta}\Big(\textstyle\sum_{j=1}^n \lambda_j e^{-\beta(t_j - t_1)}\Big)^2\right]\times\\
&\quad \times \prod_{k=2}^n \exp\left[-\frac{\sigma^2}{4\beta}\left(1 - e^{-2\beta(t_k - t_{k-1})}\right)\cdot\Big(\textstyle\sum_{j=k}^n \lambda_j e^{-\beta(t_j - t_k)}\Big)^2\right].
\end{aligned}$$

Note: the distribution of $(X_{t_1}, \dots, X_{t_n})$ depends on the difference of the consecutive epochs $t_1 < \dots < t_n$.

e) We write for all $t \geqslant 0$

$$\tilde{X}_t = e^{\beta t} X_t \quad \text{and} \quad \tilde{U}_t = e^{\beta t} U_t$$

and we show that both processes have the same finite-dimensional distributions.

Clearly, both processes are Gaussian and both have independent increments. From

$$\tilde{X}_0 = X_0 = 0 \quad \text{and} \quad \tilde{U}_0 = U_0 = 0$$

and for $s \leqslant t$

$$\begin{aligned}
\tilde{X}_t - \tilde{X}_s &= \sigma \int_s^t e^{\beta r}\, dB_r\\
&\sim \mathsf{N}\big(0,\ \tfrac{\sigma^2}{2\beta}\big(e^{2\beta t} - e^{2\beta s}\big)\big),\\
\tilde{U}_t - \tilde{U}_s &= \frac{\sigma}{\sqrt{2\beta}}\big(B(e^{2\beta t} - 1) - B(e^{2\beta s} - 1)\big)\\
&\sim \frac{\sigma}{2\beta}\, B(e^{2\beta t} - e^{2\beta s})\\
&\sim \mathsf{N}\big(0,\ \tfrac{\sigma^2}{2\beta}\big(e^{2\beta t} - e^{2\beta s}\big)\big)
\end{aligned}$$

we see that the claim is true.

Problem 18.5 (Solution) We use the time-dependent Itô formula from Problem 16.3 (or the 2-dimensional Itô-formula for the process (t, X_t)) with $f(t,x) = e^{ct}\int_0^x \frac{dy}{\sigma(y)}$. Note that the parameter c is still a free parameter.

Using Itô's multiplication rule—$(dt)^2 = dt\,dB_t = 0$ and $(dB_t)^2 = dt$ we get

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t \implies (dX_t)^2 = d\langle X\rangle_t = \sigma^2(X_t)\,dt.$$

Thus,

$$\begin{aligned}
dZ_t = df(t,X_t) &= \partial_t f(t,X_t)\,dt + \partial_x f(t,X_t)\,dX_t + \tfrac{1}{2}\,\partial_x^2 f(t,X_t)\,(dX_t)^2 \\
&= ce^{ct}\int_0^{X_t}\frac{dy}{\sigma(y)}\,dt + e^{ct}\frac{1}{\sigma(X_t)}\,dX_t - \frac{1}{2}\,e^{ct}\,\frac{\sigma'(X_t)}{\sigma^2(X_t)}\,\sigma^2(X_t)\,dt \\
&= ce^{ct}\int_0^{X_t}\frac{dy}{\sigma(y)}\,dt + e^{ct}\frac{b(X_t)}{\sigma(X_t)}\,dt + e^{ct}\,dB_t - \frac{1}{2}\,e^{ct}\,\sigma'(X_t)\,dt \\
&= e^{ct}\left[c\int_0^{X_t}\frac{dy}{\sigma(dy)} - \frac{1}{2}\,\sigma'(X_t) + \frac{b(X_t)}{\sigma(X_t)}\right]dt + e^{ct}\,dB_t.
\end{aligned}$$

Let us show that the expression in the brackets $[\cdots]$ is constant if we choose c appropriately. For this we differentiate this expression:

$$\begin{aligned}
\frac{d}{dx}\left[c\int_0^x\frac{dy}{\sigma(dy)} - \frac{1}{2}\,\sigma'(x) + \frac{b(x)}{\sigma(x)}\right] &= \frac{c}{\sigma(x)} - \frac{d}{dx}\left[\frac{1}{2}\,\sigma'(x) - \frac{b(x)}{\sigma(x)}\right] \\
&= \frac{c}{\sigma(x)} - \left[\frac{1}{2}\,\sigma''(x) - \frac{d}{dx}\,\frac{b(x)}{\sigma(x)}\right] \\
&= \frac{1}{\sigma(x)}\Bigg(c - \underbrace{\sigma(x)\left[\frac{1}{2}\,\sigma''(x) - \frac{d}{dx}\,\frac{b(x)}{\sigma(x)}\right]}_{=\text{const. by assumption}}\Bigg)
\end{aligned}$$

This shows that we should choose c in such a way that the expression $c - \sigma\cdot[\cdots]$ becomes zero, i.e.

$$c = \sigma(x)\left[\frac{1}{2}\,\sigma''(x) - \frac{d}{dx}\,\frac{b(x)}{\sigma(x)}\right].$$

Problem 18.6 (Solution) Set $f(t,x) = tx$. Then

$$\partial_t f(t,x) = x, \qquad \partial_x f(t,x) = t, \qquad \partial_x^2 f(t,x) = 0.$$

Using the time-dependent Itô formula (cf. Problem 16.3) or the 2-dimensional Itô formula (cf. Theorem 16.6) for the process (t, B_t) we get

$$\begin{aligned}
dX_t &= \partial_t f(t,B_t)\,dt + \partial_x f(t,B_t)\,dB_t + \tfrac{1}{2}\,\partial_x^2 f(t,B_t)\,dt \\
&= B_t\,dt + t\,dB_t \\
&= \frac{X_t}{t}\,dt + t\,dB_t.
\end{aligned}$$

Together with the initial condition $X_0 = 0$ this is the SDE which has $X_t = tB_t$ as solution.

The trouble is, that the solution is not unique! To see this, assume that X_t and Y_t are any two solutions. Then

$$dZ_t := d(X_t - Y_t) = dX_t - dY_t = \left(\frac{X_t}{t} - \frac{Y_t}{t}\right) dt = \frac{Z_t}{t}\,dt, \quad Z_0 = 0.$$

This is an ODE and all (deterministic) processes $Z_t = ct$ are solutions with initial condition $Z_0 = 0$. If we want to enforce uniqueness, we need a condition on Z_0'. So

$$dX_t = \frac{X_t}{t}\,dt + t\,dB_t \quad \text{and} \quad \frac{d}{dt}X_t\Big|_{t=0} = x_0'$$

will do. (Note that tB_t is differentiable at $t = 0$!).

Problem 18.7 (Solution) a) With the argument from Problem 18.6, i.e. Itô's formula, we get for $f(t,x) = x/(1+t)$

$$\partial_t f(t,x) = -\frac{x}{(1+t)^2}, \qquad \partial_x f(t,x) = \frac{1}{1+t}, \qquad \partial_x^2 f(t,x) = 0.$$

And so

$$\begin{aligned} dU_t &= -\frac{B_t}{(1+t)^2}\,dt + \frac{1}{1+t}\,dB_t \\ &= -\frac{U_t}{1+t}\,dt + \frac{1}{1+t}\,dB_t. \end{aligned}$$

The initial condition is $U_0 = 0$.

b) Using Itô's formula for $f(x) = \sin x$ we get, because of $\sin^2 x + \cos^2 x = 1$, that

$$\begin{aligned} dV_t &= \cos B_t\,dB_t - \tfrac{1}{2}\sin B_t\,dt \\ &= \sqrt{1 - \sin^2 B_t}\,dB_t - \tfrac{1}{2}\sin B_t\,dt \\ &= \sqrt{1 - V_t^2}\,dB_t - \tfrac{1}{2}V_t\,dt \end{aligned}$$

and the initial condition is $V_0 = 0$.

c) Using Itô's formula in each coordinate we get

$$\begin{aligned} d\begin{pmatrix} X_t \\ Y_t \end{pmatrix} &= \begin{pmatrix} -a\sin B_t \\ b\cos B_t \end{pmatrix} dB_t + \frac{1}{2}\begin{pmatrix} -a\cos B_t \\ -b\sin B_t \end{pmatrix} dt \\ &= \begin{pmatrix} -\frac{a}{b}\,b\sin B_t \\ \frac{b}{a}\,a\cos B_t \end{pmatrix} dB_t - \frac{1}{2}\begin{pmatrix} a\cos B_t \\ b\sin B_t \end{pmatrix} dt \\ &= \begin{pmatrix} -\frac{a}{b}\,Y_t \\ \frac{b}{a}\,X_t \end{pmatrix} dB_t - \frac{1}{2}\begin{pmatrix} X_t \\ Y_t \end{pmatrix} dt. \end{aligned}$$

The initial condition is $(X_0, Y_0) = (a, 0)$.

Problem 18.8 (Solution) a) We use Example 18.4 (and 18.3) where we set

$$\alpha(t) \equiv b, \qquad \beta(t) \equiv 0, \qquad \gamma(t) \equiv 0, \qquad \delta(t) \equiv \sigma.$$

Then we get

$$\begin{aligned} dX_t^\circ &= \sigma^2 X_t^\circ\,dt - \sigma X_t^\circ\,dB_t \\ dZ_t &= bX_t^\circ\,dt \end{aligned}$$

and, by Example 18.3 we see

$$\begin{aligned} dX_t^\circ &= X_0^\circ \exp\left(\int_0^t \left(\sigma^2 - \tfrac{1}{2}\sigma^2\right) ds - \int_0^t \sigma\,dB_s\right) \\ &= X_0^\circ \exp\left(\tfrac{1}{2}\sigma^2 t - \sigma\,B_t\right) \\ Z_t &= \int_0^t bX_s^\circ\,ds \end{aligned}$$

Thus,

$$\begin{aligned} Z_t &= \int_0^t bX_0^\circ\, e^{\frac{1}{2}\sigma^2 s - \sigma B_s}\,ds \\ X_t &= \frac{Z_t}{X_t^\circ} = be^{-\frac{1}{2}\sigma^2 t + \sigma B_t} \int_0^t e^{\frac{1}{2}\sigma^2 s - \sigma B_s}\,ds. \end{aligned}$$

We finally have to adjust the initial condition by adding $X_0 = x_0$ to the X_t we have just found:

$$\implies X_t = X_0 + be^{-\frac{1}{2}\sigma^2 t + \sigma B_t} \int_0^t e^{\frac{1}{2}\sigma^2 s - \sigma B_s}\,ds.$$

b) We use Example 18.4 (and 18.3) where we set

$$\alpha(t) \equiv m, \qquad \beta(t) \equiv -1, \qquad \gamma(t) \equiv \sigma, \qquad \delta(t) \equiv 0.$$

Then we get

$$\begin{aligned} dX_t^\circ &= X_t^\circ\,dt \\ dZ_t &= mX_t^\circ\,dt + \sigma X_t^\circ\,dB_t \end{aligned}$$

Thus,

$$\begin{aligned} X_t^\circ &= X_0^\circ\, e^t \\ Z_t &= \int_0^t m\,e^s\,ds + \sigma \int_0^t e^s\,dB_s \\ &= m\,(e^t - 1) + \sigma \int_0^t e^s\,dB_s \\ X_t &= \frac{Z_t}{X_t^\circ} = m\,(1 - e^{-t}) + \sigma \int_0^t e^{s-t}\,dB_s \end{aligned}$$

and, if we take care of the initial condition $X_0 = x_0$, we get

$$\implies X_t = x_0 + m\,(1 - e^{-t}) + \sigma \int_0^t e^{s-t}\,dB_s.$$

Problem 18.9 (Solution) Set

$$b(x) = \sqrt{1 + x^2} + \tfrac{1}{2}\,x \quad \text{and} \quad \sigma(x) = \sqrt{1 + x^2}.$$

Then we get (using the notation of Lemma 18.8)

$$\sigma'(x) = \frac{x}{\sqrt{1+x^2}} \quad \text{and} \quad \kappa(x) = \frac{b(x)}{\sigma(x)} - \tfrac{1}{2}\sigma'(x) = 1.$$

Using the Ansatz of Lemma 18.8 we set

$$d(x) = \int_0^x \frac{dy}{\sigma(y)} = \operatorname{arsinh} x \quad \text{and} \quad Z_t = f(X_t) = d(X_t).$$

Using Itô's formula gives

$$\begin{aligned}
dZ_t &= \partial_x f(X_t)\,dX_t + \tfrac{1}{2}\partial_x^2 f(X_t)\,\sigma^2(X_t)\,dt \\
&= \frac{1}{\sigma(X_t)}\,dX_t + \tfrac{1}{2}\left(\tfrac{1}{\sigma}\right)'(X_t)\,\sigma^2(X_t)\,dt \\
&= \left(1 + \frac{X_t}{2\sqrt{1+X_t^2}}\right)dt + dB_t + \tfrac{1}{2}\left(-\frac{X_t}{(1+X_t^2)^{3/2}}\right)\left(1+X_t^2\right)dt \\
&= dt + dB_t,
\end{aligned}$$

and so $Z_t = Z_0 + t + B_t$. Finally,

$$X_t = \sinh(Z_0 + t + B_t) \quad \text{where} \quad Z_0 = \operatorname{arsinh} X_0.$$

Problem 18.10 (Solution) Set $b = b(t,x), b_0 = b(t,0)$ etc. Observe that $\|b\| = \left(\sum_j |b_j(t,x)|^2\right)^{1/2}$ and $\|\sigma\| = \left(\sum_{j,k} |\sigma_{jk}(t,x)|^2\right)^{1/2}$ are norms; therefore, we get using the triangle estimate and the elementary inequality $(a+b)^2 \leqslant 2(a^2+b^2)$

$$\begin{aligned}
\|b\|^2 + \|\sigma\|^2 &= \|b - b_0 + b_0\|^2 + \|\sigma - \sigma_0 + \sigma_0\|^2 \\
&\leqslant 2\|b-b_0\|^2 + 2\|\sigma-\sigma_0\|^2 + 2\|b_0\|^2 + 2\|\sigma_0\|^2 \\
&\leqslant 2L^2|x|^2 + 2\|b_0\|^2 + 2\|\sigma_0\|^2 \\
&\leqslant 2L^2(1+|x|)^2 + 2(\|b_0\|^2 + \|\sigma_0\|^2)(1+|x|)^2 \\
&\leqslant 2\left(L^2 + \|b_0\|^2 + \|\sigma_0\|^2\right)(1+|x|)^2.
\end{aligned}$$

Problem 18.11 (Solution) a) If $b(x) = -e^x$ and $X_0^x = x$ we have to solve the following ODE/integral equation

$$X_t^x = x - \int_0^t e^{X_s^x}\,ds$$

and it is not hard to see that the solution is

$$X_t^x = \log\left(\frac{1}{t+e^{-x}}\right).$$

This shows that

$$\lim_{x\to\infty} X_t^x = \lim_{x\to\infty} \log\left(\frac{1}{t+e^{-x}}\right) = \log\frac{1}{t} = -\log t.$$

This means that Corollary 18.21 fails in this case since the coefficient of the ODE grows too fast.

b) Now assume that $|b(x)| + |\sigma(x)| \leqslant M$ for all x. Then we have

$$\left|\int_0^t b(X_s)\,ds\right| \leqslant Mt.$$

By Itô's isometry we get

$$\mathbb{E}\left[\left|\int_0^t \sigma(X_s^x)\,dB_s\right|^2\right] = \mathbb{E}\left[\int_0^t |\sigma^2(X_s^x)|\,ds\right] \leqslant M^2 t.$$

Using $(a+b)^2 \leqslant 2a^2 + 2b^2$ we see

$$\begin{aligned}
\mathbb{E}(|X_t^x - x|^2) &\leqslant 2\,\mathbb{E}\left[\left|\int_0^t b(X_s)\,ds\right|^2\right] + 2\,\mathbb{E}\left[\left|\int_0^t \sigma(X_s^x)\,dB_s\right|^2\right] \\
&\leqslant 2(Mt)^2 + 2M^2 t \\
&= 2M^2 t(t+1).
\end{aligned}$$

By Fatou's lemma

$$\mathbb{E}\left(\varliminf_{|x|\to\infty} |X_t^x - x|^2\right) \leqslant \varliminf_{|x|\to\infty} \mathbb{E}(|X_t^x - x|^2) \leqslant 2M^2 t(t+1)$$

which shows that $|X_t^x|$ cannot be bounded as $|x| \to \infty$.

c) Assume now that $b(x)$ and $\sigma(x)$ grow like $|x|^{p/2}$ for some $p \in (0,2)$. A calculation as above yields

$$\left|\int_0^t b(X_s)\,ds\right|^2 \underset{\text{Schwarz}}{\overset{\text{Cauchy}}{\leqslant}} t\int_0^t |b(X_s)|^2\,ds \leqslant c_p t\int_0^t (1+|X_s|^p)\,ds$$

and, by Itô's isometry

$$\mathbb{E}\left[\left|\int_0^t \sigma(X_s^x)\,dB_s\right|^2\right] = \mathbb{E}\left[\int_0^t |\sigma^2(X_s^x)|\,ds\right] \leqslant c' \int_0^t \mathbb{E}(1+|X_s|^p)\,ds.$$

Using $(a+b)^2 \leqslant 2a^2 + 2b^2$ and Theorem 18.18 we get

$$\begin{aligned}
\mathbb{E}\,|X_t^x - x|^2 &\leqslant 2c_p t\int_0^t \big(1+\mathbb{E}(|X_s|^p)\big)\,ds + 2c'\int_0^t \big(1+\mathbb{E}(|X_s|^p)\big)\,ds \\
&\leqslant c_{t,p} + c'_{t,p}\int_0^t |x|^p\,dt \\
&= c_{t,p} + t\,c'_{t,p}|x|^p.
\end{aligned}$$

Again by Fatou's theorem we see that the left-hand side grows like $|x|^2$ (if X_t^x is unbounded) while the (larger!) right-hand side grows like $|x|^p$, $p < 2$, and this is impossible.

Thus, $(X_t^x)_x$ is unbounded as $|x| \to \infty$.

Problem 18.12 (Solution) We have to show

$$\frac{|x-y|}{(1+|x|)(1+|y|)} \overset{!}{\leqslant} \left|\frac{x}{|x|^2} - \frac{y}{|y|^2}\right|$$

$$\begin{aligned}
\iff && \frac{|x-y|^2}{(1+|x|)^2(1+|y|)^2} &\leqslant \left|\frac{x}{|x|^2}-\frac{y}{|y|^2}\right|^2\\
\iff && \frac{|x|^2-2\langle x,y\rangle+|y|^2}{(1+|x|)^2(1+|y|)^2} &\leqslant \frac{|x|^2}{|x|^4}-\frac{2\langle x,y\rangle}{|x|^2|y|^2}+\frac{|y|^2}{|y|^4}\\
\iff && 2\langle x,y\rangle\left(\frac{1}{|x|^2|y|^2}-\frac{1}{(1+|x|)^2(1+|y|)^2}\right) &\leqslant \frac{1}{|x|^2}+\frac{1}{|y|^2}-\frac{|x|^2+|y|^2}{(1+|x|)^2(1+|y|)^2}\\
\iff && 2\langle x,y\rangle\left(\frac{1}{|x|^2|y|^2}-\frac{1}{(1+|x|)^2(1+|y|)^2}\right) &\leqslant \left(|x|^2+|y|^2\right)\left(\frac{1}{|x|^2|y|^2}-\frac{1}{(1+|x|)^2(1+|y|)^2}\right).
\end{aligned}$$

By the Cauchy-Schwarz inequality we get $2\langle x,y\rangle\leqslant 2|x|\cdot|y|\leqslant|x|^2+|y|^2$, and this shows that the last estimate is correct.

19 On Diffusions

Problem 19.1 (Solution) We have

$$Au = Lu = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}\partial_i\partial_j u + \sum_{i=1}^{d} b_i\partial_i u$$

and we know that $L : \mathcal{C}_c^\infty \to \mathcal{C}$. Fix $R > 0$ and $i, j \in \{1, \ldots, d\}$ where $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ such that $\chi|_{\mathbb{B}(0,R)} \equiv 1$.

For all $u, \chi \in \mathcal{C}^2$ we get

$$\begin{aligned}
L(\phi u) &= \frac{1}{2}\sum_{i,j} a_{ij}\partial_i\partial_j(\phi u) + \sum_i b_i\partial_i(\phi u) \\
&= \frac{1}{2}\sum_{i,j} a_{ij}\big(\partial_i\partial_j\phi + \partial_i\partial_j u + \partial_i\phi\partial_j u + \partial_i u\partial_j\phi\big) + \sum_i b_i\big(u\partial_i\phi + \phi\partial_i u\big) \\
&= \phi Lu + uL\phi + \sum_{i,j} a_{ij}\partial_i\phi\partial_j u
\end{aligned}$$

where we used the symmetry $a_{ij} = a_{ji}$ in the last step.

Now use $u(x) = x_i$ and $\phi(x) = \chi(x)$. Then $u\chi \in \mathcal{C}_c^\infty$, $L(u\chi) \in \mathcal{C}$ and so

$$L(u\chi)(x) = b_i(x) \qquad \text{for all } |x| < R \implies b_i|_{\mathbb{B}(0,R)} \text{ continuous.}$$

Now use $u(x) = x_i x_j$ and $\phi(x) = \chi(x)$. Then $u\chi \in \mathcal{C}_c^\infty$, $L(u\chi) \in \mathcal{C}$ and so

$$L(u\chi)(x) = a_{ij} + x_j b_i(x) + x_i b_j(x) \qquad \text{for all } |x| < R \implies a_{ij}|_{\mathbb{B}(0,R)} \text{ continuous.}$$

Since $R > 0$ is arbitrary, the claim follows.

Problem 19.2 (Solution) This is a straightforward application of the differentiation Lemma which is familiar from measure and integration theory, cf. Schilling [11, Theorem 11.5, pp. 92–93]: observe that by our assumptions

$$\left|\frac{\partial^2 p(t,x,y)}{\partial x_j\partial x_k}\right| \leqslant C(t) \quad \text{for all } x, y \in \mathbb{R}^d$$

which shows that for $u \in C_c^\infty(\mathbb{R}^d)$

$$\left|\frac{\partial^2 p(t,x,y)}{\partial x_j\partial x_k}\, u(y)\right| \leqslant C(t)\,|u(y)| \in L^1(\mathbb{R}^d) \tag{*}$$

for each $t > 0$. Thus we get

$$\frac{\partial^2}{\partial x_j\partial x_k}\int p(t,x,y)\,u(y)\,dy = \int \frac{\partial^2}{\partial x_j\partial x_k}\, p(t,x,y)\,u(y)\,dy.$$

Moreover, (*) and the fact that $p(t,\cdot,y) \in \mathcal{C}_\infty(\mathbb{R}^d)$ allow us to change limits and integrals to get for $x \to x_0$ and $|x| \to \infty$

$$
\begin{aligned}
\lim_{x\to x_0} \int \frac{\partial^2}{\partial x_j \partial x_k} p(t,x,y)\,u(y)\,dy &= \int \lim_{x\to x_0} \frac{\partial^2}{\partial x_j \partial x_k} p(t,x,y)\,u(y)\,dy \\
&= \int \frac{\partial^2}{\partial x_j \partial x_k} p(t,x_0,y)\,u(y)\,dy \\
&\Longrightarrow T_t \text{ maps } \mathcal{C}_c^\infty(\mathbb{R}^d) \text{ into } \mathcal{C}(\mathbb{R}^d); \\
\lim_{|x|\to \infty} \int \frac{\partial^2}{\partial x_j \partial x_k} p(t,x,y)\,u(y)\,dy &= \int \underbrace{\lim_{|x|\to\infty} \frac{\partial^2}{\partial x_j \partial x_k} p(t,x,y)}_{=0}\,u(y)\,dy = 0 \\
&\Longrightarrow T_t \text{ maps } \mathcal{C}_c^\infty(\mathbb{R}^d) \text{ into } \mathcal{C}_\infty(\mathbb{R}^d).
\end{aligned}
$$

Addition: With a standard uniform boundedness and density argument we can show that T_t maps $\mathcal{C}_\infty$ into $\mathcal{C}_\infty$: fix $u \in \mathcal{C}_\infty(\mathbb{R}^d)$ and pick a sequence $(u_n)_n \subset \mathcal{C}_c^\infty(\mathbb{R}^d)$ such that

$$\lim_{n\to\infty} \|u - u_n\|_\infty = 0.$$

Then we get

$$\|T_t u - T_t u_n\|_\infty = \|T_t(u-u_n)\|_\infty \leqslant \|u-u_n\|_\infty \xrightarrow[n\to\infty]{} 0$$

which means that $T_t u_n \to T_t u$ uniformly, i.e. $T_t u \in \mathcal{C}_\infty$ as $T_t u_n \in \mathcal{C}_\infty$.

Problem 19.3 (Solution) Let $u \in \mathcal{C}_\infty^2$. Then there is a sequence of test functions $(u_n)_n \subset \mathcal{C}_c^\infty$ such that $\|u_n - u\|_{(2)} \to 0$. Thus, $u_n \to u$ uniformly and $A(u_n - u_m) \to 0$ uniformly. The closedness now gives $u \in \mathfrak{D}(A)$.

Problem 19.4 (Solution) Let $u, \phi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. Then

$$
\begin{aligned}
\langle Lu, \phi\rangle_{L^2} &= \sum_{i,j} \int_{\mathbb{R}^d} a_{ij}\partial_i\partial_j u \cdot \phi\,dx + \sum_j \int_{\mathbb{R}^d} b_j \partial_j u \cdot \phi\,dx + \int_{\mathbb{R}^d} cu \cdot \phi\,dx \\
&\overset{\substack{\text{int by}\\ \text{parts}}}{=} \sum_{i,j} \int_{\mathbb{R}^d} u \cdot \partial_i\partial_j(a_{ij}\phi)\,dx - \sum_j \int_{\mathbb{R}^d} u \cdot \partial_j(b_j\phi)\,dx + \int_{\mathbb{R}^d} u \cdot c\phi\,dx \\
&= \langle u, L^*\phi\rangle_{L^2}
\end{aligned}
$$

where

$$L^*(x,D_x)\phi(x) = \sum_{ij} \partial_i\partial_j\big(a_{ij}(x)\phi(x)\big) - \sum_j \partial_j\big(b_j(x)\phi(x)\big) + c(x)\phi(x).$$

Now assume that we are in $(t,x) \in [0,\infty) \times \mathbb{R}^d$—the case $\mathbb{R} \times \mathbb{R}^d$ is easier, as we have no boundary term. Consider $L + \partial_t = L(x,D_x) + \partial_t$ for sufficiently smooth $u = u(t,x)$ and $\phi = \phi(t,x)$ with compact support in $[0,\infty) \times \mathbb{R}^d$. We find

$$
\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^d} (L+\partial_t)u(t,x) \cdot \phi(t,x)\,dx\,dt \\
&= \int_0^\infty \int_{\mathbb{R}^d} Lu(t,x) \cdot \phi(t,x)\,dx\,dt + \int_0^\infty \int_{\mathbb{R}^d} \partial_t u(t,x) \cdot \phi(t,x)\,dx\,dt
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^\infty \int_{\mathbb{R}^d} Lu(t,x)\cdot\phi(t,x)\,dx\,dt + \int_{\mathbb{R}^d}\int_0^\infty \partial_t u(t,x)\cdot\phi(t,x)\,dt\,dx \\
&= \int_0^\infty \int_{\mathbb{R}^d} u(t,x)\cdot L^*\phi(t,x)\,dx\,dt + \int_{\mathbb{R}^d}\Big(u(t,x)\phi(t,x)\Big|_{t=0}^\infty - \int_0^\infty u(t,x)\cdot\partial_t\phi(t,x)\,dt\Big)\,dx \\
&= \int_0^\infty \int_{\mathbb{R}^d} u(t,x)\cdot L^*\phi(t,x)\,dx\,dt - \int_{\mathbb{R}^d}\Big(u(0,x)\phi(0,x) + \int_0^\infty u(t,x)\cdot\partial_t\phi(t,x)\,dt\Big)\,dx.
\end{aligned}
$$

This shows that $(L(x,D_x)+\partial_t)^* = L^*(x,D_x) - \partial_t - \delta_{(0,x)}$.

Problem 19.5 (Solution) Using Lemma 7.10 we get for all $u \in \mathcal{C}_c^\infty(\mathbb{R}^d)$

$$
\begin{aligned}
& \frac{d}{dt}T_t u(x) = T_t L(\cdot,D)u(x) \\
\Longrightarrow\quad & \frac{d}{dt}\int p(t,x,y)\,u(y)\,dy = \int p(t,x,y)\,L(y,D_y)u(y)\,dy \\
\Longrightarrow\quad & \int \frac{d}{dt}p(t,x,y)\,u(y)\,dy = \int p(t,x,y)\,L(y,D_y)u(y)\,dy.
\end{aligned}
$$

The change of differentiation and integration can easily be justified by a routine application of the differentiation lemma (e.g. Schilling [11, Theorem 11.5, pp. 92–93]): under our assumptions we have for all $\epsilon \in (0,1)$ and $R > 0$

$$
\sup_{t\in[\epsilon,1/\epsilon]}\sup_{|x|\leqslant R}\left|\frac{d}{dt}p(t,x,y)\,u(y)\right| \leqslant C(\epsilon,R)\,|u(y)| \in L^1(\mathbb{R}^d).
$$

Inserting the expression for the differential operator $L(y,D_y)$, we find for the right-hand side

$$
\begin{aligned}
&\int p(t,x,y)\,L(y,D_y)u(y)\,dy \\
&= \frac{1}{2}\sum_{j,k=1}^d \int p(t,x,y)\cdot a_{jk}(y)\frac{\partial^2 u(y)}{\partial y_j\partial y_k}\,dy + \sum_{j=1}^d \int p(t,x,y)\cdot b_j(y)\frac{\partial u(y)}{\partial y_j}\,dy \\
&\overset{\text{int. by parts}}{=} \frac{1}{2}\sum_{j,k=1}^d \int \frac{\partial^2}{\partial y_j\partial y_k}\Big(a_{jk}(y)\cdot p(t,x,y)\Big)u(y)\,dy + \sum_{j=1}^d \int \frac{\partial}{\partial y_j}\Big(b_j(y)\cdot p(t,x,y)\Big)u(y)\,dy \\
&= \int L^*(y,D_y)p(t,x,y)\,u(y)\,dy
\end{aligned}
$$

and the claim follows since $u \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ is arbitrary.

Problem 19.6 (Solution) Problem 6.2 shows that X_t is a Markov process. The continuity of the sample paths is obvious and so is the Feller property (using the form of the transition function found in the solution of Problem 6.2).

Let us calculate the generator. Set $I_t = \int_0^t B_s\,ds$. The semigroup is given by

$$
T_t u(x,y) = \mathbb{E}^{x,y}u(B_t,I_t) = \mathbb{E}\,u\left(B_t+x,\ \textstyle\int_0^t(B_s+x)\,ds+y\right) = \mathbb{E}\,u(B_t+x, I_t+tx+y).
$$

If we differentiate the expression under the expectation with respect to t, we get with the help of Itô's formula

$$
\begin{aligned}
du(B_t+x,I_t+tx+y) &= \partial_x u(B_t+x,I_t+tx+y)\,dB_t \\
&\quad + \partial_y u(B_t+x,I_t+tx+y)\,d(I_t+tx)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{2}\,\partial_x^2 u(B_t+x, I_t+tx+y)\,dt\\
&= \partial_x u(B_t+x, I_t+tx+y)\,dB_t\\
&\quad + \partial_y u(B_t+x, I_t+tx+y)(B_t+x)\,dt\\
&\quad + \frac{1}{2}\,\partial_x^2 u(B_t+x, I_t+tx+y)\,dt
\end{aligned}
$$

since $dB_s\,dI_s = 0$. So,

$$
\begin{aligned}
\mathbb{E}\,u(B_t+x, I_t+tx+y) - u(x,y) &= \int_0^t \mathbb{E}\left[\partial_y u(B_s+x, I_s+sx+y)(B_s+x)\right]ds\\
&\quad + \frac{1}{2}\int_0^t \mathbb{E}\left[\partial_x^2 u(B_s+x, I_s+sx+y)\right]ds.
\end{aligned}
$$

Dividing by t and letting $t\to 0$ we get

$$
Lu(x,y) = x\,\partial_y u(x,y) + \frac{1}{2}\,\partial_x^2 u(x,y).
$$

Problem 19.7 (Solution) We assume for a) and b) that the operator L is more general than written in (19.1), namely

$$
Lu(x) = \frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\,\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{j=1}^d b_j(x)\,\frac{\partial u(x)}{\partial x_j} + c(x)u(x)
$$

where all coefficients are continuous functions.

a) If u has compact support, then Lu has compact support. Since, by assumption, the coefficients of L are continuous, Lu is bounded, hence M_t^u is square integrable.

Obviously, M_t^u is $\mathcal{F}_t$ measurable. Let us establish the martingale property. For this we fix $s\leqslant t$. Then

$$
\begin{aligned}
\mathbb{E}^x\left(M_t^u \,\middle|\, \mathcal{F}_s\right) &= \mathbb{E}^x\left(u(X_t) - u(X_0) - \int_0^t Lu(X_r)\,dr \,\middle|\, \mathcal{F}_s\right)\\
&= \mathbb{E}^x\left(u(X_t) - u(X_s) - \int_s^t Lu(X_r)\,dr \,\middle|\, \mathcal{F}_s\right)\\
&\qquad + u(X_s) - u(X_0) - \int_0^s Lu(X_r)\,dr\\
&= \mathbb{E}^x\left(u(X_t) - u(X_s) - \int_0^{t-s} Lu(X_{r+s})\,dr \,\middle|\, \mathcal{F}_s\right) + M_s^u\\
&\underset{\text{property}}{\overset{\text{Markov}}{=}} \mathbb{E}^{X_s}\left(u(X_{t-s}) - u(X_0) - \int_0^{t-s} Lu(X_r)\,dr\right) + M_s^u.
\end{aligned}
$$

Observe that $T_t u(y) = \mathbb{E}^y\,u(X_t)$ is the semigroup associated with the Markov process. Then

$$
\begin{aligned}
&\mathbb{E}^y\left(u(X_{t-s}) - u(X_0) - \int_0^{t-s} Lu(X_r)\,dr\right)\\
&\qquad = T_{t-s}u(y) - u(y) - \int_0^{t-s} \mathbb{E}^y\left(Lu(X_r)\right)dr = 0
\end{aligned}
$$

by Lemma 7.10, see also Theorem 7.21. This shows that $\mathbb{E}^x\left(M_t^u \,\middle|\, \mathcal{F}_s\right) = M_s^u$, and we are done.

b) Fix $R > 0$, $x \in \mathbb{R}^d$, and pick a smooth cut-off function $\chi = \chi_R \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ such that $\chi|\mathbb{B}(x,R) \equiv 1$. Then for all $f \in \mathcal{C}^2(\mathbb{R}^d)$ we have $\chi f \in \mathcal{C}_c^2(\mathbb{R}^d)$ and it is not hard to see that the calculation in part a) still holds for such functions.

Set $\tau = \tau_R^x = \inf\{t > 0 : |X_t - x| \geqslant R\}$. This is a stopping time and we have

$$f(X_t^\tau) = \chi(X_t^\tau)f(X_t^\tau) = (\chi f)(X_t^\tau).$$

Moreover,

$$\begin{aligned}
L(\chi f) &= \frac{1}{2}\sum_{i,j} a_{ij}\partial_i\partial_j(\chi f) + \sum_i b_i\partial_i(\chi f) + c\chi f \\
&= \frac{1}{2}\sum_{i,j} a_{ij}\big(f\partial_i\partial_j\chi + \chi\partial_i\partial_j f + \partial_i\chi\partial_j f + \partial_i f\partial_j\chi\big) + \sum_i b_i\big(f\partial_i\chi + \chi\partial_i f\big) + c\chi f \\
&= \chi Lf + fL\chi + \sum_{i,j} a_{ij}\partial_i\chi\partial_j f - c\chi f
\end{aligned}$$

where we used the symmetry $a_{ij} = a_{ji}$ in the last step.

This calculation shows that $L(\chi f) = Lf$ on $\mathbb{B}(x,R)$.

By optional stopping and part a) we know that $(M_{t\wedge\tau_R}^{\chi f}, \mathcal{F}_t)_{t\geqslant 0}$ is a martingale. Moreover, we get for $s \leqslant t$

$$\begin{aligned}
\mathbb{E}^x\left(M_{t\wedge\tau_R}^f \,\middle|\, \mathcal{F}_s\right) &= \mathbb{E}^x\left(M_{t\wedge\tau_R}^{\chi f} \,\middle|\, \mathcal{F}_s\right) \\
&= M_{s\wedge\tau_R}^{\chi f} \\
&= M_{s\wedge\tau_R}^f.
\end{aligned}$$

Since $(\tau_R)_R$ is a localizing sequence, we are done.

c) A diffusion operator L satisfies that $c = 0$. Thus, the calculation for $L(\chi f)$ in part b) shows that

$$L(u\phi) - uL\phi - \phi Lu = \sum_{ij} a_{ij}\partial_i u\partial_j\phi = \nabla u(x)\cdot a(x)\nabla\phi(x).$$

This proves the second equality in the formula of the problem.

For the first we note that $d\langle M^u, M^\phi\rangle_t = dM_t^u dM_t^\phi$ (by the definition of the bracket process) and the latter we can calculate with the rules for Itô differentials. We have

$$dX_t^j = \sum_k \sigma_{jk}(X_t)dB_t^k + b_j(X_t)\,dt$$

and, by Itô's formula,

$$du(X_t) = \sum_j \partial_j u(X_t)\,dX_t^j + dt\text{-terms} = \sum_{j,k}\partial_j u(X_t)\sigma_{jk}(X_t)\,dB_t^k + dt\text{-terms}.$$

By definition,

$$dM_t^u = du(X_t) - Lu(X_t)\,dt = \sum_{j,k}\partial_j u(X_t)\sigma_{jk}(X_t)\,dB_t^k + dt\text{-terms}.$$

Thus, using that all terms containing $(dt)^2$ and $dB_t^k dt$ are zero, we get

$$
\begin{aligned}
dM_t^u\,dM_t^\phi &= \sum_{j,k}\sum_{l,m}\partial_j u(X_t)\partial_l\phi(X_t)\sigma_{jk}(X_t)\sigma_{lm}(X_t)\,dB_t^k\,dB_t^m \\
&= \sum_{j,k}\sum_{l,m}\partial_j u(X_t)\partial_l\phi(X_t)\sigma_{jk}(X_t)\sigma_{lm}(X_t)\,\delta_{km}\,dt \\
&= \sum_{j,l}\partial_j u(X_t)\partial_l\phi(X_t)\sum_k\sigma_{jk}(X_t)\sigma_{lk}(X_t)\;dt \\
&= \sum_{j,l}\partial_j u(X_t)\partial_l\phi(X_t)a_{jl}\;dt \\
&= \nabla u(X_t)\cdot a(X_t)\nabla\phi(X_t)
\end{aligned}
$$

where $a_{jl} = \sum_k \sigma_{jk}(X_t)\sigma_{lk}(X_t) = (\sigma\sigma^\top)_{jl}$. ($x\cdot y$ denotes the Euclidean scalar product and $\nabla = (\partial_1,\dots,\partial_d)^\top$.)

Bibliography

[1] Chung, K.L.: *Lectures from Markov Processes to Brownian Motion.* Springer, New York 1982. Enlarged and reprinted as [2].

[2] Chung, K.L., Walsh, J.B.: *Markov Processes, Brownian Motion, and Time Symmetry (Second Edition).* Springer, Berlin 2005. This is the 2nd edn. of [1].

[3] Dupuis, C.: Mesure de Hausdorff de la trajectoire de certains processus à accroissements indépendants et stationnaires. In: *Séminaire de Prohabilités*, Springer, Lecture Notes in Mathematics **381**, Berlin 1974, 40–77.

[4] Gradshteyn, I.S., Ryzhik, I.M.: *Table of Integrals, Series, and Products (4th corrected and enlarged edn).* Academic Press, San Diego 1980.

[5] Karatzas, I., Shreve, S.E.: *Brownian Motion and Stochastic Calculus.* Springer, New York 1988.

[6] Pazy, A.: *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Springer, New York 1983.

[7] Protter, P.E.: *Stochastic Integration and Differential Equations (Second Edn.).* Springer, Berlin 2004.

[8] A. Rényi: *Probability Theory.* Dover, Mineola (NY) 2007. (Reprint of the 1970 edition published by Norh-Holland and Akadémiai Kiadó.)

[9] Rudin, W.: *Principles of Mathematical Analysis.* McGraw-Hill, Auckland 1976 (3rd edn).

[10] Rudin, W.: *Real and Complex Analysis.* McGraw-Hill, New York 1986 (3rd edn).

[11] Schilling, R.L.: *Measures, Integrals and Martingales.* Cambridge University Press, Cambridge 2011 (3rd printing with corrections).

[12] Schilling, R.L. and Partzsch, L.: *Brownian Motion. An Introduction to Stochastic Processes.* De Gruyter, Berlin 2012.

[13] Trèves, F.: *Topological Vector Spaces, Distributions and Kernels.* Academic Press, San Diego (CA) 1990.