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Solution Manual Chapter 1–12

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1 Prologue.
Solutions to Problems 1.1–1.2

Problem 1.1 Name the figures on the left and right *Figure 1* and *Figure 2*, respectively. Figure 1 is a triangle but Figure 2 is a (convex) quadrangle: the 'hypotenuse' has a kink. This is easily seen by comparing in Figure 2 the slopes of the small triangle in the lower left (it is 2/5) and the larger triangle on top (it is $3/8 \neq 2/5$).

Problem 1.2 We have to calculate the area of an isosceles triangle of side-length r, base b, height h and opening angle $\phi := 2\pi/2^j$. From elementary geometry we know that

$$\cos\tfrac{\phi}{2} = \tfrac{h}{r} \quad \text{and} \quad \sin\tfrac{\phi}{2} = \tfrac{b}{2r}$$

so that

$$\text{area (triangle)} = \frac{1}{2}\,hb = r^2 \cos\tfrac{\phi}{2}\sin\tfrac{\phi}{2} = \frac{r^2}{2}\,\sin\phi\,.$$

Since we have $\lim_{\phi\to 0}\frac{\sin\phi}{\phi} = 1$ we find

$$\begin{aligned}\text{area (circle)} &= \lim_{j\to\infty} 2^j\,\frac{r^2}{2}\,\sin\tfrac{2\pi}{2^j}\\ &= 2r^2\pi\lim_{j\to\infty}\frac{\sin\frac{2\pi}{2^j}}{\frac{2\pi}{2^j}}\\ &= 2r^2\pi\end{aligned}$$

just as we had expected.

2 The pleasures of counting. Solutions to Problems 2.1–2.21

Problem 2.1 (i) We have

$$\begin{aligned} x \in A \setminus B &\iff x \in A \text{ and } x \notin B \\ &\iff x \in A \text{ and } x \in B^c \\ &\iff x \in A \cap B^c. \end{aligned}$$

(ii) Using (i) and de Morgan's laws (*) yields

$$\begin{aligned} (A \setminus B) \setminus C &\overset{\text{(i)}}{=} (A \cap B^c) \cap C^c = A \cap B^c \cap C^c \\ &= A \cap (B^c \cap C^c) \overset{(*)}{=} A \cap (B \cup C)^c = A \setminus (B \cup C). \end{aligned}$$

(iii) Using (i), de Morgan's laws (*) and the fact that $(C^c)^c = C$ gives

$$\begin{aligned} A \setminus (B \setminus C) &\overset{\text{(i)}}{=} A \cap (B \cap C^c)^c \\ &\overset{(*)}{=} A \cap (B^c \cup C) \\ &= (A \cap B^c) \cup (A \cap C) \\ &\overset{\text{(i)}}{=} (A \setminus B) \cup (A \cap C). \end{aligned}$$

(iv) Using (i) and de Morgan's laws (*) gives

$$\begin{aligned} A \setminus (B \cap C) &\overset{\text{(i)}}{=} A \cap (B \cap C)^c \\ &\overset{(*)}{=} A \cap (B^c \cup C^c) \\ &= (A \cap B^c) \cup (A \cap C^c) \\ &\overset{\text{(i)}}{=} (A \setminus B) \cup (A \setminus C) \end{aligned}$$

(v) Using (i) and de Morgan's laws (*) gives

$$\begin{aligned} A \setminus (B \cup C) &\overset{\text{(i)}}{=} A \cap (B \cup C)^c \\ &\overset{(*)}{=} A \cap (B^c \cap C^c) \\ &= A \cap B^c \cap C^c \\ &= A \cap B^c \cap A \cap C^c \\ &\overset{\text{(i)}}{=} (A \setminus B) \cup (A \setminus C) \end{aligned}$$

Problem 2.2 Observe, first of all, that

$$A \setminus C \subset (A \setminus B) \cup (B \setminus C). \tag{$*$}$$

This follows easily from

$$\begin{aligned} A \setminus C &= (A \setminus C) \cap X \\ &= (A \cap C^c) \cap (B \cup B^c) \\ &= (A \cap C^c \cap B) \cup (A \cap C^c \cap B^c) \\ &\subset (B \cap C^c) \cup (A \cap B^c) \\ &= (B \setminus C) \cup (A \setminus B). \end{aligned}$$

Using this and the analogous formula for $C \setminus A$ then gives

$$\begin{aligned} &(A \cup B \cup C) \setminus (A \cap B \cap C) \\ &\quad= (A \cup B \cup C) \cap (A \cap B \cap C)^c \\ &\quad= [A \cap (A \cap B \cap C)^c] \cup [B \cap (A \cap B \cap C)^c] \cup [C \cap (A \cap B \cap C)^c] \\ &\quad= [A \setminus (A \cap B \cap C)] \cup [B \setminus (A \cap B \cap C)] \cup [C \setminus (A \cap B \cap C)] \\ &\quad= [A \setminus (B \cap C)] \cup [B \setminus (A \cap C)] \cup [C \setminus (A \cap B)] \\ &\overset{(2.1(\text{iv}))}{=} (A \setminus B) \cup (A \setminus C) \cup (B \setminus A) \cup (B \setminus C) \cup (C \setminus A) \cup (C \setminus B) \\ &\overset{(*)}{=} (A \setminus B) \cup (B \setminus A) \cup (B \setminus C) \cup (C \setminus B) \\ &\quad= (A \Delta B) \cup (B \Delta C) \end{aligned}$$

Problem 2.3 It is clearly enough to prove (2.3) as (2.2) follows if I contains 2 points. De Morgan's identities state that for any index set I (finite, countable or not countable) and any collection of subsets $A_i \subset X$, $i \in I$, we have

$$\text{(a)} \quad \left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} A_i^c \qquad \text{and} \qquad \text{(b)} \quad \left(\bigcap_{i\in I} A_i\right)^c = \bigcup_{i\in I} A_i^c.$$

In order to see (a) we note that

$$\begin{aligned} a \in \left(\bigcup_{i\in I} A_i\right)^c &\iff a \notin \bigcup_{i\in I} A_i \\ &\iff \forall\, i \in I \,:\, a \notin A_i \\ &\iff \forall\, i \in I \,:\, a \in A_i^c \\ &\iff a \in \bigcap_{i\in I} A_i^c, \end{aligned}$$

and (b) follows from

$$a \in \left(\bigcap_{i\in I} A_i\right)^c \iff a \notin \bigcap_{i\in I} A_i$$

$$\iff \exists\, i_0 \in I \,:\, a \notin A_{i_0}$$
$$\iff \exists\, i_0 \in I \,:\, a \in A_{i_0}^c$$
$$\iff a \in \bigcup_{i \in I} A_i^c.$$

Problem 2.4 (i) The inclusion $f(A \cap B) \subset f(A) \cap f(B)$ is *always* true since $A \cap B \subset A$ and $A \cap B \subset B$ imply that $f(A \cap B) \subset f(A)$ and $f(A \cap B) \subset f(B)$, respectively. Thus, $f(A \cap B) \subset f(A) \cap f(B)$.

Furthermore, $y \in f(A) \setminus f(B)$ means that there is some $x \in A$ but $x \notin B$ such that $y = f(x)$, that is: $y \in f(A \setminus B)$. Thus, $f(A) \setminus f(B) \subset f(A \setminus B)$.

To see that the converse inclusions cannot hold we consider some *non injective* f. Take $X = [0,2]$, $A = (0,1)$, $B = (1,2)$, and $f : [0,2] \to \mathbb{R}$ with $x \mapsto f(x) = c$ (c is some constant). Then f is not injective and

$$\emptyset = f(\emptyset) = f((0,1) \cap (1,2)) \neq f((0,1)) \cup f((1,2)) = \{c\}.$$

Moreover, $f(X) = f(B) = \{c\} = f(X \setminus B)$ but $f(X) \setminus f(B) = \emptyset$.

(ii) Recall, first of all, the definition of f^{-1} for a map $f : X \to Y$ and $B \subset Y$

$$f^{-1}(B) := \{x \in X \,:\, f(x) \in B\}.$$

Observe that

$$\begin{aligned} x \in f^{-1}(\cup_{i\in I} C_i) &\iff f(x) \in \cup_{i\in I} C_i \\ &\iff \exists\, i_0 \in I \,:\, f(x) \in C_{i_0} \\ &\iff \exists\, i_0 \in I \,:\, x \in f^{-1}(C_{i_0}) \\ &\iff x \in \cup_{i\in I} f^{-1}(C_i), \end{aligned}$$

and

$$\begin{aligned} x \in f^{-1}(\cap_{i\in I} C_i) &\iff f(x) \in \cap_{i\in I} C_i \\ &\iff \forall\, i \in I \,:\, f(x) \in C_i \\ &\iff \forall\, i \in I \,:\, x \in f^{-1}(C_i) \\ &\iff x \in \cap_{i\in I} f^{-1}(C_i), \end{aligned}$$

and, finally,

$$\begin{aligned} x \in f^{-1}(C \setminus D) &\iff f(x) \in C \setminus D \\ &\iff f(x) \in C \quad \text{and} \quad f(x) \notin D \\ &\iff x \in f^{-1}(C) \quad \text{and} \quad x \notin f^{-1}(D) \\ &\iff x \in f^{-1}(C) \setminus f^{-1}(D). \end{aligned}$$

Problem 2.5

(i), (vi) For every x we have

$$\begin{aligned}\mathbb{1}_{A\cap B}(x) = 1 &\iff x \in A \cap B\\ &\iff x \in A,\ x \in B\\ &\iff \mathbb{1}_A(x) = 1 = \mathbb{1}_B(x)\\ &\iff \begin{cases}\mathbb{1}_A(x)\cdot\mathbb{1}_B(x) = 1\\ \min\{\mathbb{1}_A(x), \mathbb{1}_B(x)\} = 1\end{cases}\end{aligned}$$

(ii), (v) For every x we have

$$\begin{aligned}\mathbb{1}_{A\cup B}(x) = 1 &\iff x \in A \cup B\\ &\iff x \in A \text{ or } x \in B\\ &\iff \mathbb{1}_A(x) + \mathbb{1}_B(x) \geqslant 1\\ &\iff \begin{cases}\min\{\mathbb{1}_A(x) + \mathbb{1}_B(x), 1\} = 1\\ \max\{\mathbb{1}_A(x), \mathbb{1}_B(x)\} = 1\end{cases}\end{aligned}$$

(iii) Since $A = (A \cap B) \cupdot (A \setminus B)$ we see that $\mathbb{1}_{A\cap B}(x) + \mathbb{1}_{A\setminus B}(x)$ can never have the value 2, thus part (ii) implies

$$\begin{aligned}\mathbb{1}_A(x) = \mathbb{1}_{(A\cap B)\cupdot(A\setminus B)}(x) &= \min\{\mathbb{1}_{A\cap B}(x) + \mathbb{1}_{A\setminus B}(x), 1\}\\ &= \mathbb{1}_{A\cap B}(x) + \mathbb{1}_{A\setminus B}(x)\end{aligned}$$

and all we have to do is to subtract $\mathbb{1}_{A\cap B}(x)$ on both sides of the equation.

(iv) With the same argument that we used in (iii) and with the result of (iii) we get

$$\begin{aligned}\mathbb{1}_{A\cup B}(x) &= \mathbb{1}_{(A\setminus B)\cupdot(A\cap B)\cupdot(B\setminus A)}(x)\\ &= \mathbb{1}_{A\setminus B}(x) + \mathbb{1}_{A\cap B}(x) + \mathbb{1}_{B\setminus A}(x)\\ &= \mathbb{1}_A(x) - \mathbb{1}_{A\cap B}(x) + \mathbb{1}_{A\cap B}(x) + \mathbb{1}_B(x) - \mathbb{1}_{A\cap B}(x)\\ &= \mathbb{1}_A(x) + \mathbb{1}_B(x) - \mathbb{1}_{A\cap B}(x).\end{aligned}$$

Problem 2.6 (i) Using 2.5(iii), (iv) we see that

$$\begin{aligned}\mathbb{1}_{A\,\triangle\,B}(x) &= \mathbb{1}_{(A\setminus B)\cupdot(B\setminus A)}(x)\\ &= \mathbb{1}_{A\setminus B}(x) + \mathbb{1}_{B\setminus A}(x)\\ &= \mathbb{1}_A(x) - \mathbb{1}_{A\cap B}(x) + \mathbb{1}_B(x) - \mathbb{1}_{A\cap B}(x)\\ &= \mathbb{1}_A(x) + \mathbb{1}_B(x) - 2\mathbb{1}_{A\cap B}(x)\end{aligned}$$

and this expression is 1 if, and only if, x is either in A or B but not in both sets. Thus

$$\mathbb{1}_{A\,\triangle\,B}(x) \iff \mathbb{1}_A(x) + \mathbb{1}_B(x) = 1 \iff \mathbb{1}_A(x) + \mathbb{1}_B(x) \text{mod} 2 = 1.$$

It is also possible to show that

$$\mathbb{1}_{A \,\triangle\, B} = |\mathbb{1}_A - \mathbb{1}_B|.$$

This follows from

$$\mathbb{1}_A(x) - \mathbb{1}_B(x) = \begin{cases} 0, & \text{if } \ x \in A \cap B; \\ 0, & \text{if } \ x \in A^c \cap B^c; \\ +1, & \text{if } \ x \in A \setminus B; \\ -1, & \text{if } \ x \in B \setminus A. \end{cases}$$

Thus,

$$|\mathbb{1}_A(x) - \mathbb{1}_B(x)| = 1 \iff x \in (A \setminus B) \cup (B \setminus A) = A \,\triangle\, B.$$

(ii) From part (i) we see that

$$\begin{aligned} \mathbb{1}_{A \,\triangle\, (B \,\triangle\, C)} &= \mathbb{1}_A + \mathbb{1}_{B \,\triangle\, C} - 2\mathbb{1}_A \mathbb{1}_{B \,\triangle\, C} \\ &= \mathbb{1}_A + \mathbb{1}_B + \mathbb{1}_C - 2\mathbb{1}_B\mathbb{1}_C - 2\mathbb{1}_A\big(\mathbb{1}_B + \mathbb{1}_C - 2\mathbb{1}_B\mathbb{1}_C\big) \\ &= \mathbb{1}_A + \mathbb{1}_B + \mathbb{1}_C - 2\mathbb{1}_B\mathbb{1}_C - 2\mathbb{1}_A\mathbb{1}_B - 2\mathbb{1}_A\mathbb{1}_C + 4\mathbb{1}_A\mathbb{1}_B\mathbb{1}_C \end{aligned}$$

and this expression treats A, B, C in a completely symmetric way, i.e.

$$\mathbb{1}_{A \,\triangle\, (B \,\triangle\, C)} = \mathbb{1}_{(A \,\triangle\, B) \,\triangle\, C}.$$

(iii) **Step 1:** $(\mathcal{P}(X), \triangle, \emptyset)$ is an abelian group.

Neutral element: $A \,\triangle\, \emptyset = \emptyset \,\triangle\, A = A$;

Inverse element: $A \,\triangle\, A = (A \setminus A) \cup (A \setminus A) = \emptyset$, i.e. each element is its own inverse.

Associativity: see part (ii);

Commutativity: $A \,\triangle\, B = B \,\triangle\, A$.

Step 2: For the multiplication $\cap$ we have

Associativity: $A \cap (B \cap C) = (A \cap B) \cap C$;

Commutativity: $A \cap B = B \cap A$;

One-element: $A \cap X = X \cap A = A$.

Step 3: Distributive law:

$$A \cap (B \,\triangle\, C) = (A \cap B) \,\triangle\, (A \cap C).$$

For this we use again indicator functions and the rules from (i) and Problem 2.5:

$$\begin{aligned} \mathbb{1}_{A \cap (B \,\triangle\, C)} = \mathbb{1}_A \mathbb{1}_{B \,\triangle\, C} &= \mathbb{1}_A(\mathbb{1}_B + \mathbb{1}_C \mod 2) \\ &= \big[\mathbb{1}_A(\mathbb{1}_B + \mathbb{1}_C)\big] \mod 2 \end{aligned}$$

$$= \big[\mathbb{1}_A \mathbb{1}_B + \mathbb{1}_A \mathbb{1}_C\big] \mod 2$$
$$= \big[\mathbb{1}_{A\cap B} + \mathbb{1}_{A\cap C}\big] \mod 2$$
$$= \mathbb{1}_{(A\cap B)\,\triangle\,(A\cap C)}.$$

Problem 2.7 Let $f : X \to Y$. One has

$$f \text{ surjective} \iff \forall\, B \subset Y \,:\, f\circ f^{-1}(B) = B$$
$$\iff \forall\, B \subset Y \,:\, f\circ f^{-1}(B) \supset B.$$

This can be seen as follows: by definition $f^{-1}(B) = \{x \,:\, f(x) \in B\}$ so that

$$f\circ f^{-1}(B) = f\big(\{x \,:\, f(x)\in B\}\big) = \{f(x) \,:\, f(x)\in B\} \subset \{y \,:\, y \in B\}$$

and we have equality in the last step if, and only if, we can guarantee that every $y \in B$ is of the form $y = f(x)$ for some x. Since this must hold for all sets B, this amounts to saying that $f(X) = Y$, i.e. that f is surjective. The second equivalence is clear since our argument shows that the inclusion '$\subset$' always holds.

Thus, we can construct a counterexample by setting $f : \mathbb{R} \to \mathbb{R}$, $f(x) := x^2$ and $B = [-1,1]$. Then

$$f^{-1}([-1,1]) = [0,1] \quad\text{and}\quad f\circ f^{-1}([-1,1]) = f([0,1]) = [0,1] \subsetneq [-1,1].$$

On the other hand

$$f \text{ injective} \iff \forall\, A \subset X \,:\, f^{-1}\circ f(A) = A$$
$$\iff \forall\, A \subset X \,:\, f^{-1}\circ f(A) \subset A.$$

To see this we observe that because of the definition of f^{-1}

$$f^{-1}\circ f(A) = \{x \,:\, f(x) \in f(A)\} \supset \{x \,:\, x \in A\} = A \tag{$*$}$$

since $x \in A$ always entails $f(x) \in f(A)$. The reverse is, for non-injective f, wrong since then there might be some $x_0 \notin A$ but with $f(x_0) = f(x) \in f(A)$ i.e. $x_0 \in f^{-1}\circ f(A) \setminus A$. This means that we have equality in $(*)$ if, and only if, f is injective. The second equivalence is clear since our argument shows that the inclusion '$\supset$' always holds.

Thus, we can construct a counterexample by setting $f : \mathbb{R} \to \mathbb{R}$, $f \equiv 1$. Then

$$f([0,1]) = \{1\} \quad\text{and}\quad f^{-1}\circ f([0,1]) = f^{-1}(\{1\}) = \mathbb{R} \supsetneq [0,1].$$

Problem 2.8 Assume that for x, y we have $f\circ g(x) = f\circ g(y)$. Since f is injective, we conclude that

$$f(g(x)) = f(g(y)) \implies g(x) = g(y),$$

and, since g is also injective,

$$g(x) = g(y) \implies x = y$$

showing that $f\circ g$ is injective.

Problem 2.9 • Call the set of odd numbers $\mathcal{O}$. Every odd number is of the form $2k-1$ where $k \in \mathbb{N}$. We are done, if we can show that the map $f : \mathbb{N} \to \mathcal{O}$, $k \mapsto 2k-1$ is bijective. Surjectivity is clear as $f(\mathbb{N}) = \mathcal{O}$. For injectivity we take $i, j \in \mathbb{N}$ such that $f(i) = f(j)$. The latter means that $2i - 1 = 2j - 1$, so $i = j$, i.e. injectivity.

- The quickest solution is to observe that $\mathbb{N} \times \mathbb{Z} = \mathbb{N} \times \mathbb{N} \cup \mathbb{N} \times \{0\} \cup \mathbb{N} \times (-\mathbb{N})$ where $-\mathbb{N} := \{-n : n \in \mathbb{N}\}$ are the strictly negative integers. We know from Example 2.5(iv) that $\mathbb{N} \times \mathbb{N}$ is countable. Moreover, the map $\beta : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \times (-\mathbb{N})$, $\beta((i,k)) = (i,-k)$ is bijective, thus $\#\mathbb{N} \times (-\mathbb{N}) = \#\mathbb{N} \times \mathbb{N}$ is also countable and so is $\mathbb{N} \times \{0\}$ since $\gamma : \mathbb{N} \to \mathbb{N} \times \{0\}$, $\gamma(n) := (n, 0)$ is also bijective.

  Therefore, $\mathbb{N} \times \mathbb{Z}$ is a union of three countable sets, hence countable.

  An *alternative approach* would be to write out $\mathbb{Z} \times \mathbb{N}$ (the swap of $\mathbb{Z}$ and $\mathbb{N}$ is for notational reasons—since the map $\beta((j,k)) := (k,j)$ from $\mathbb{Z} \times \mathbb{N}$ to $\mathbb{N} \times \mathbb{Z}$ is bijective, the cardinality does not change) in the following form

$$\begin{array}{ccccccccc}
\dots & (-3,1) & (-2,1) & (-1,1) & (0,1) & (1,1) & (2,1) & (3,1) & \dots \\
\dots & (-3,2) & (-2,2) & (-1,2) & (0,2) & (1,2) & (2,2) & (3,2) & \dots \\
\dots & (-3,3) & (-2,3) & (-1,3) & (0,3) & (1,3) & (2,3) & (3,3) & \dots \\
\dots & (-3,4) & (-2,4) & (-1,4) & (0,4) & (1,4) & (2,4) & (3,4) & \dots \\
\dots & (-3,5) & (-2,5) & (-1,5) & (0,5) & (1,5) & (2,5) & (3,5) & \dots \\
\dots & (-3,6) & (-2,6) & (-1,6) & (0,6) & (1,6) & (2,6) & (3,6) & \dots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

  and going through the array, starting with $(0,1)$, then $(1,1) \to (1,2) \to (0,2) \to (-1,2) \to (-1,1)$, then $(2,1) \to (2,2) \to (2,3) \to (1,3) \to \dots$ in clockwise oriented $\sqcup$-shapes down, left, up.

- In Example 2.5(iv) we have shown that $\#\mathbb{Q} \leqslant \#\mathbb{N}$. Since $\mathbb{N} \subset \mathbb{Q}$, we have a canonical injection $\jmath : \mathbb{N} \to \mathbb{Q}, i \mapsto i$ so that $\#\mathbb{N} \leqslant \#\mathbb{Q}$. Using Theorem 2.7 we conclude that $\#\mathbb{Q} = \#\mathbb{N}$.

  The proof of $\#(\mathbb{N} \times \mathbb{N}) = \#\mathbb{N}$ can be easily adapted—using some pretty obvious notational changes—to show that the Cartesian product of any two countable sets of cardinality $\#\mathbb{N}$ has again cardinality $\#\mathbb{N}$. Applying this $m-1$ times we see that $\#\mathbb{Q}^n = \#\mathbb{N}$.

- $\bigcup_{m \in \mathbb{N}} \mathbb{Q}^m$ is a countable union of countable sets, hence countable, cf. Theorem 2.6.

Problem 2.10 Following the hint it is clear that $\beta : \mathbb{N} \to \mathbb{N} \times \{1\}, i \mapsto (i, 1)$ is a bijection and that $\jmath : \mathbb{N} \times \{1\} \to \mathbb{N} \times \mathbb{N}, (i,1) \mapsto (i,1)$ is an injection. Thus, $\#\mathbb{N} \leqslant \#(\mathbb{N} \times \mathbb{N})$.

On the other hand, $\mathbb{N} \times \mathbb{N} = \bigcup_{j \in \mathbb{N}} \mathbb{N} \times \{j\}$ which is a countable union of countable sets, thus $\#(\mathbb{N} \times \mathbb{N}) \leqslant \#\mathbb{N}$.

Applying Theorem 2.7 finally gives $\#(\mathbb{N} \times \mathbb{N}) = \#\mathbb{N}$.

Problem 2.11 Since $E \subset F$ the map $\jmath : E \to F, e \mapsto e$ is an injection, thus $\#E \leqslant \#F$.

Problem 2.12 Assume that the set $\{0,1\}^{\mathbb{N}}$ were indeed countable and that $\{s_j\}_{j\in\mathbb{N}}$ was an enumeration: each s_j would be a sequence of the form $(d_1^j, d_2^j, d_3^j, ..., d_k^j, ...)$ with $d_k^j \in \{0,1\}$. We could write these sequences in an infinite list of the form:

$$\begin{array}{ccccccccc}
s_1 & = & d_1^1 & d_2^1 & d_3^1 & d_4^1 & \ldots & d_k^1 & \ldots \\
s_2 & = & d_1^2 & d_2^2 & d_3^2 & d_4^2 & \ldots & d_k^2 & \ldots \\
s_3 & = & d_1^3 & d_2^3 & d_3^3 & d_4^3 & \ldots & d_k^3 & \ldots \\
s_4 & = & d_1^4 & d_2^4 & d_3^4 & d_4^4 & \ldots & d_k^4 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\
s_k & = & d_1^k & d_2^k & d_3^k & d_4^k & \ldots & d_k^k & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots
\end{array}$$

and produce a new 0-1-sequence $S = (e_1, e_2, e_3, \ldots)$ by setting

$$e_m := \begin{cases} 0, & \text{if} \quad d_m^m = 1 \\ 1, & \text{if} \quad d_m^m = 0 \end{cases} .$$

Since S differs from s_ℓ exactly at position ℓ, S cannot be in the above list, thus, the above list did not contain all 0-1-sequences, hence a contradiction.

Problem 2.13 Consider the function $f : (0,1) \to \mathbb{R}$ given by

$$f(x) := \frac{1}{1-x} - \frac{1}{x}.$$

This function is obviously continuous and we have $\lim_{x\to 0} f(x) = -\infty$ and $\lim_{x\to 1} f(x) = +\infty$. By the intermediate value theorem we have therefore $f((0,1)) = \mathbb{R}$, i.e. surjectivity. Since f is also differentiable and $f'(x) = \dfrac{1}{(1-x)^2} + \dfrac{1}{x^2} > 0$, we see that f is strictly increasing, hence injective, hence bijective.

Problem 2.14 Since $A_1 \subset \bigcup_{j\in\mathbb{N}} A_j$ it is clear that $\mathfrak{c} = \#A_1 \leqslant \#\bigcup_{j\in\mathbb{N}} A_j$. On the other hand, $\#A_j = \mathfrak{c}$ means that we can map A_j bijectively onto $\mathbb{R}$ and, using Problem 2.13, we map $\mathbb{R}$ bijectively onto $(0,1)$ or $(j-1,j)$. This shows that $\#\bigcup_{j\in\mathbb{N}} A_j \leqslant \#\bigcup_{j\in\mathbb{N}}(j-1,j) \leqslant \#\mathbb{R} = \mathfrak{c}$. Using Theorem 2.7 finishes the proof.

Problem 2.15 Since we can write each $x \in (0,1)$ as an infinite dyadic fraction (o.k. if it is finite, fill it up with an infinite tail of zeroes !), the proof of Theorem 2.8 shows that $\#(0,1) \leqslant \#\{0,1\}^{\mathbb{N}}$.

On the other hand, thinking in base-4 expansions, each element of $\{1,2\}^{\mathbb{N}}$ can be interpreted as a unique base-4 fraction (having no 0 or 3 in its expansion) of some number in $(0,1)$. Thus, $\#\{1,2\}^{\mathbb{N}} \leqslant \#\mathbb{N}$.

But $\#\{1,2\}^{\mathbb{N}} = \#\{0,1\}^{\mathbb{N}}$ and we conclude with Theorem 2.7 that $\#(0,1) = \#\{0,1\}^{\mathbb{N}}$.

Problem 2.16 Just as before, expand $x \in (0,1)$ as an n-adic fraction, then interpret each element of $\{1,2,\dots,n+1\}^{\mathbb{N}}$ as a unique $(n+1)$-adic expansion of a number in $(0,1)$ and observe that $\#\{1,2,\dots,n+1\}^{\mathbb{N}} = \{0,1,\dots,n\}^{\mathbb{N}}$.

Problem 2.17 Take a vector $(x,y) \in (0,1)\times(0,1)$ and expand its coordinate entries x,y as dyadic numbers:

$$x = 0.x_1x_2x_3\dots, \qquad y = 0.y_1y_2y_3\dots.$$

Then $z := 0.x_1y_1x_2y_2x_3y_3\dots$ is a number in $(0,1)$. Conversely, we can 'zip' each $z = 0.z_1z_2z_3z_4\dots \in (0,1)$ into two numbers $x,y \in (0,1)$ by setting

$$x := 0.z_2z_4z_6z_8\dots, \qquad y := 0.z_1z_3z_5z_7\dots$$

This is obviously a bijective operation.

Since we have a bijection between $(0,1) \leftrightarrow \mathbb{R}$ it is clear that we have also a bijection between $(0,1)\times(0,1) \leftrightarrow \mathbb{R}\times\mathbb{R}$.

Problem 2.18 We have seen in Problem 2.18 that $\#\{0,1\}^{\mathbb{N}} = \#\{1,2\}^{\mathbb{N}} = \mathfrak{c}$. Obviously, $\{1,2\}^{\mathbb{N}} \subset \mathbb{N}^{\mathbb{N}} \subset \mathbb{R}^{\mathbb{N}}$ and since we have a bijection between $(0,1) \leftrightarrow \mathbb{R}$ one extends this (using coordinates) to a bijection between $(0,1)^{\mathbb{N}} \leftrightarrow \mathbb{R}^{\mathbb{N}}$. Using Theorem 2.9 we get

$$\mathfrak{c} = \#\{1,2\}^{\mathbb{N}} \leqslant \#\mathbb{N}^{\mathbb{N}} \leqslant \#\mathbb{R}^{\mathbb{N}} = \mathfrak{c},$$

and, because of Theorem 2.7 we have equality in the above formula.

Problem 2.19 Let $F \in \mathcal{F}$ with $\#F = n$ Then we can write F as a tuple of length n (having n pairwise different entries...) and therefore we can interpret F as an element of $\bigcup_{m\in\mathbb{N}} \mathbb{N}^m$. In this sense, $\mathcal{F} \hookrightarrow \bigcup_{m\in\mathbb{N}} \mathbb{N}^m$ and $\#\mathcal{F} \leqslant \bigcup_{m\in\mathbb{N}} \mathbb{N}^m = \#\mathbb{N}$ since countably many countable sets are again countable. Since $\mathbb{N} \subset \mathcal{F}$ we get $\#\mathcal{F} = \#\mathbb{N}$ by Theorem 2.7.

Alternative: Define a map $\phi : \mathcal{F} \to \mathbb{N}$ by

$$\mathcal{F} \ni A \mapsto \phi(A) := \sum_{a\in A} 2^a$$

. It is clear that ϕ increases if A gets bigger: $A \subset B \implies \phi(A) \leqslant \phi(B)$. Let $A,B \in \mathcal{F}$ be two finite sets, say $A = \{a_1,a_2,\dots,a_M\}$ and $\{b_1,b_2,\dots,b_N\}$ (ordered according to size with a_1,b_1 being the smallest and a_M,b_N the biggest) such that $\phi(A) = \phi(B)$. Assume, to the contrary, that $A \neq B$. If $a_M \neq b_N$, say $a_M > b_N$, then

$$\begin{aligned}\phi(A) \geqslant \phi(\{a_M\}) \geqslant 2^{a_M} > \frac{2^{a_M}-1}{2-1} &= \sum_{j=1}^{a_M-1} 2^j \\ &= \phi(\{1,2,3,\dots a_M-1\}) \\ &\geqslant \phi(B),\end{aligned}$$

which cannot be the case since we assumed $\phi(A) = \phi(B)$. Thus, $a_M = b_N$. Now consider recursively the next elements, a_{M-1} and b_{N-1} and the same conclusion yields their equality etc. The process stops after $\min\{M, N\}$ steps. But if $M \neq N$, say $M > N$, then A would contain at least one more element than B, hence $\phi(A) > \phi(B)$, which is also a contradiction. This, finally shows that $A = B$, hence that ϕ is injective.

On the other hand, each natural number can be expressed in terms of finite sums of powers of base-2, so that ϕ is also surjective.

Thus, $\#\mathcal{F} = \#\mathbb{N}$.

Problem 2.20 (Let $\mathcal{F}$ be as in the previous exercise.) Observe that the infinite sets from $\mathcal{P}(\mathbb{N})$, $\mathcal{I} := \mathcal{P}(\mathbb{N}) \setminus \mathcal{F}$ can be surjectively mapped onto $\{0,1\}^{\mathbb{N}}$: if $\{a_1, a_2, a_3, \dots\} = A \subset \mathbb{N}$, then define an infinite 0-1-sequence $(b_1, b_2, b_3, \dots)$ by setting $b_j = 0$ or $b_j = 1$ according to whether a_j is even or odd. This is a surjection of $\mathcal{P}(\mathbb{N})$ onto $\{0,1\}^{\mathbb{N}}$ and so $\#\mathcal{P}(\mathbb{N}) \geqslant \#\{0,1\}^{\mathbb{N}}$. Call this map γ and consider the family $\gamma^{-1}(s)$, $s \in \{0,1\}^{\mathbb{N}}$ in $\mathcal{I}$, consisting of obviously disjoint infinite subsets of $\mathbb{N}$ which lead to the same 0-1-sequence s. Now choose from each family $\gamma^{-1}(s)$ a representative, call it $r(s) \in \mathcal{I}$. Then the map $s \mapsto r(s)$ is a bijection between $\{0,1\}^{\mathbb{N}}$ and a subset of $\mathcal{I}$, the set of all representatives. Hence, $\mathcal{I}$ has at least the same cardinality as $\{0,1\}^{\mathbb{N}}$ and as such a bigger cardinality than $\mathbb{N}$.

Problem 2.21 Denote by Θ the map $\mathcal{P}(\mathbb{N}) \ni A \mapsto \mathbb{1}_A \in \{0,1\}^{\mathbb{N}}$. Let $\delta = (d_1, d_2, d_3, \dots) \in \{0,1\}^{\mathbb{N}}$ and define $A(\delta) := \{j \in \mathbb{N} : d_j = 1\}$. Then $\delta = (\mathbb{1}_{A(\delta)}(j))_{j\in\mathbb{N}}$ showing that Θ is surjective.

On the other hand,

$$\mathbb{1}_A = \mathbb{1}_B \iff \mathbb{1}_A(j) = \mathbb{1}_B(j) \;\; \forall j \in \mathbb{N} \iff A = B.$$

This shows the injectivity of Θ, and $\#\mathcal{P}(\mathbb{N}) = \#\{0,1\}^{\mathbb{N}}$ follows.

3 σ-Algebras.
Solutions to Problems 3.1–3.12

Problem 3.1 (i) It is clearly enough to show that $A, B \in \mathcal{A} \implies A \cap B \in \mathcal{A}$, because the case of N sets follows from this by induction, the induction step being

$$\underbrace{A_1 \cap \ldots \cap A_N}_{=:B \in \mathcal{A}} \cap A_{N+1} = B \cap A_{N+1} \in \mathcal{A}.$$

Let $A, B \in \mathcal{A}$. Then, by (Σ_2) also $A^c, B^c \in \mathcal{A}$ and, by (Σ_3) and (Σ_2)

$$A \cap B = (A^c \cup B^c)^c = (A^c \cup B^c \cup \emptyset \cup \emptyset \cup \ldots)^c \in \mathcal{A}.$$

Alternative: Of course, the last argument also goes through for N sets:

$$\begin{aligned} A_1 \cap A_2 \cap \ldots \cap A_N &= (A_1^c \cup A_2^c \cup \ldots \cup A_N^c)^c \\ &= (A_1^c \cup \ldots \cup A_N^c \cup \emptyset \cup \emptyset \cup \ldots)^c \in \mathcal{A}. \end{aligned}$$

(ii) By (Σ_2) we have $A \in \mathcal{A} \implies A^c \in \mathcal{A}$. Use A^c instead of A and observe that $(A^c)^c = A$ to see the claim.

(iii) Clearly $A^c, B^c \in \mathcal{A}$ and so, by part (i), $A \setminus B = A \cap B^c \in \mathcal{A}$ as well as $A \triangle B = (A \setminus B) \cup (B \setminus A) \in \mathcal{A}$.

Problem 3.2 (iv) Let us assume that $B \neq \emptyset$ and $B \neq X$. Then $B^c \notin \{\emptyset, B, X\}$. Since with B also B^c must be contained in a σ-algebra, the family $\{\emptyset, B, X\}$ cannot be one.

(vi) Set $\mathcal{A}_E := \{E \cap A : A \in \mathcal{A}\}$. The key observation is that all set operations in $\mathcal{A}_E$ are now relative to E and not to X. This concerns mainly the complementation of sets! Let us check (Σ_1)–(Σ_3).

Clearly $\emptyset = E \cap \emptyset \in \mathcal{A}_E$. If $B \in \mathcal{A}$, then $B = E \cap A$ for some $A \in \mathcal{A}$ and the complement of B relative to E is $E \setminus B = E \cap B^c = E \cap (E \cap A)^c = E \cap (E^c \cup A^c) = E \cap A^c \in \mathcal{A}_E$ as $A^c \in \mathcal{A}$. Finally, let $(B_j)_{j \in \mathbb{N}} \subset \mathcal{A}_E$. Then there are $(A_j)_{j \in \mathbb{N}} \subset \mathcal{A}$ such that $B_j = E \cap A_j$. Since $A = \bigcup_{j \in \mathbb{N}} A_j \in \mathcal{A}$ we get $\bigcup_{j \in \mathbb{N}} B_j = \bigcup_{j \in \mathbb{N}} (E \cap A_j) = E \cap \bigcup_{j \in \mathbb{N}} A_j = E \cap A \in \mathcal{A}_E$.

(vii) Note that f^{-1} interchanges with all set operations. Let $A, A_j, j \in \mathbb{N}$ be sets in $\mathcal{A}$. We know that then $A = f^{-1}(A'), A_j = f^{-1}(A_j')$ for suitable $A, A_j' \in \mathcal{A}'$. Since $\mathcal{A}'$ is, by assumption a σ-algebra, we have

$$\emptyset = f^{-1}(\emptyset) \in \mathcal{A} \qquad\qquad \text{as} \quad \emptyset \in \mathcal{A}'$$

$$A^c = \left(f^{-1}(A')\right)^c = f^{-1}(A'^c) \in \mathcal{A} \qquad \text{as} \quad A'^c \in \mathcal{A}'$$

$$\bigcup_{j\in\mathbb{N}} A_j = \bigcup_{j\in\mathbb{N}} f^{-1}(A'_j) = f^{-1}\left(\bigcup_{j\in\mathbb{N}} A'_j\right) \in \mathcal{A} \qquad \text{as} \quad \bigcup_{j\in\mathbb{N}} A'_j \in \mathcal{A}'$$

which proves (Σ_1)–(Σ_3) for $\mathcal{A}$.

Problem 3.3 (i) Since $\mathcal{G}$ is a σ-algebra, $\mathcal{G}$ 'competes' in the intersection of all σ-algebras $\mathcal{C} \supset \mathcal{G}$ appearing in the definition of $\mathcal{A}$ in the proof of Theorem 3.4(ii). Thus, $\mathcal{G} \supset \sigma(\mathcal{G})$ while $\mathcal{G} \subset \sigma(\mathcal{G})$ is always true.

(ii) Without loss of generality we can assume that $\varnothing \neq A \neq X$ since this would simplify the problem. Clearly $\{\varnothing, A, A^c, X\}$ is a σ-algebra containing A and no element can be removed without losing this property. Thus $\{\varnothing, A, A^c, X\}$ is minimal and, therefore, $= \sigma(\{A\})$.

(iii) Assume that $\mathcal{F} \subset \mathcal{G}$. Then we have $\mathcal{F} \subset \mathcal{G} \subset \sigma(\mathcal{G})$. Now $\mathcal{C} := \sigma(\mathcal{G})$ is a potential 'competitor' in the intersection appearing in the proof of Theorem 3.4(ii), and as such $\mathcal{C} \supset \sigma(\mathcal{F})$, i.e. $\sigma(\mathcal{G}) \supset \sigma(\mathcal{F})$.

Problem 3.4 (i) $\{\varnothing, (0, \frac{1}{2}), \{0\} \cup [\frac{1}{2}, 1], [0, 1]\}$. We have 2 *atoms* (see the explanations below): $(0, \frac{1}{2}), (0, \frac{1}{2})^c$.

(ii) $\{\varnothing, [0, \frac{1}{4}), [\frac{1}{4}, \frac{3}{4}], (\frac{3}{4}, 1], [0, \frac{3}{4}], [\frac{1}{4}, 1], [0, \frac{1}{4}) \cup (\frac{3}{4}, 1], [0, 1]\}$. We have 3 *atoms* (see below): $[0, \frac{1}{4}), [\frac{1}{4}, \frac{3}{4}], (\frac{3}{4}, 1]$.

(iii) —same solution as (ii)—

Parts (ii) and (iii) are quite tedious to do and they illustrate how difficult it can be to find a σ-algebra containing two distinct sets.... imagine how to deal with something that is generated by 10, 20, or infinitely many sets. Instead of giving a particular answer, let us describe the method to find $\sigma(\{A, B\})$ practically, and then we are going to prove it.

1. Start with trivial sets and given sets: $\varnothing, X, A, B$.
2. now add their complements: A^c, B^c
3. now add their unions and intersections and differences: $A \cup B, A \cap B, A \smallsetminus B, B \smallsetminus A$
4. now add the complements of the sets in 3.: $A^c \cap B^c, A^c \cup B^c, (A \smallsetminus B)^c, (B \smallsetminus A)^c$
5. finally, add unions of differences and their complements: $(A \smallsetminus B) \cup (B \smallsetminus A), (A \smallsetminus B)^c \cap (B \smallsetminus A)^c$.

All in all one should have 16 sets (some of them could be empty or X or appear several times, depending on how much A differs from B). That's it, but the trouble is: is this construction correct? Here is a somewhat more systematic procedure:

Definition: An *atom* of a σ-algebra $\mathcal{A}$ is a non-void set $\varnothing \neq A \in \mathcal{A}$ that contains no other set of $\mathcal{A}$.

Since $\mathcal{A}$ is stable under intersections, it is also clear that all atoms are disjoint sets! Now we can make up every set from $\mathcal{A}$ as union (finite or countable) of such atoms. The task at hand is to find atoms if A, B are given. This is easy: the atoms of our future σ-algebra must be: $A \setminus B$, $B \setminus A$, $A \cap B$, $(A \cup B)^c$. (Test it: if you make a picture, this is a tesselation of our space X using disjoint sets and we can get back A, B as union! It is also minimal, since these sets must appear in $\sigma(\{A, B\})$ anyway.) The crucial point is now:

Theorem. *If $\mathcal{A}$ is a σ-algebra with N atoms (finitely many!), then $\mathcal{A}$ consists of exactly 2^N elements.*

Proof. The question is how many different unions we can make out of N sets. Simple answer: we find $\binom{N}{j}$, $0 \leqslant j \leqslant N$ different unions involving exactly j sets ($j = 0$ will, of course, produce the empty set) and they are all different as the atoms were disjoint. Thus, we get $\sum_{j=0}^{N} \binom{N}{j} = (1+1)^N = 2^N$ different sets.

It is clear that they constitute a σ-algebra. ∎

This answers the above question. The number of atoms depends obviously on the relative position of A, B: do they intersect, are they disjoint etc. Have fun with the exercises and do not try to find σ-algebras generated by three or more sets..... (By the way: can you think of a situation in $[0,1]$ with two subsets given and exactly *four* atoms? Can there be more?)

Problem 3.5 (i) See the solution to Problem 3.4.

(ii) If $A_1, \dots, A_N \subset X$ are given, there are at most 2^N atoms. This can be seen by induction. If $N = 1$, then there are $\#\{A, A^c\} = 2$ atoms. If we add a further set A_{N+1}, then the worst case would be that A_{N+1} intersects with each of the 2^N atoms, thus splitting each atom into two sets which amounts to saying that there are $2 \cdot 2^N = 2^{N+1}$ atoms.

Problem 3.6 $\mathcal{O}_1$ Since $\varnothing$ contains no element, every element $x \in \varnothing$ admits certainly some neighbourhood $B_\delta(x)$ and so $\varnothing \in \mathcal{O}$. Since for all $x \in \mathbb{R}^n$ also $B_\delta(x) \subset \mathbb{R}^n$, $\mathbb{R}^n$ is clearly open.

$\mathcal{O}_2$ Let $U, V \in \mathcal{O}$. If $U \cap V = \varnothing$, we are done. Else, we find some $x \in U \cap V$. Since U, V are open, we find some $\delta_1, \delta_2 > 0$ such that $B_{\delta_1}(x) \subset U$ and $B_{\delta_2}(x) \subset V$. But then we can take $h := \min\{\delta_1, \delta_2\} > 0$ and find

$$B_h(x) \subset B_{\delta_1}(x) \cap B_{\delta_2}(x) \subset U \cap V,$$

i.e. $U \cap V \in \mathcal{O}$. For finitely many, say N, sets, the same argument works. Notice that already for countably many sets we will get a problem as the radius $h := \min\{\delta_j : j \in \mathbb{N}\}$ is not necessarily any longer > 0.

$\mathcal{O}_2$ Let I be any (finite, countable, not countable) index set and $(U_i)_{i\in I} \subset \mathcal{O}$ be a family of open sets. Set $U := \bigcup_{i\in I} U_i$. For $x \in U$ we find some $j \in I$ with $x \in U_j$, and since U_j was open, we find some $\delta_j > 0$ such that $B_{\delta_j}(x) \subset U_j$. But then, trivially, $B_{\delta_j}(x) \subset U_j \subset \bigcup_{i\in I} U_i = U$ proving that U is open.

The family $\mathcal{O}^n$ cannot be a σ-algebra since the complement of an open set $U \neq \emptyset, \neq \mathbb{R}^n$ is closed.

Problem 3.7 Let $X = \mathbb{R}$ and set $U_k := (-\frac{1}{k}, \frac{1}{k})$ which is an open set. Then $\bigcap_{k\in\mathbb{N}} U_k = \{0\}$ but a singleton like $\{0\}$ is closed and not open.

Problem 3.8 We know already that the Borel sets $\mathcal{B} = \mathcal{B}(\mathbb{R})$ are generated by any of the following systems:

$$\{[a,b) : a,b \in \mathbb{Q}\},\ \{[a,b) : a,b \in \mathbb{R}\},$$
$$\{(a,b) : a,b \in \mathbb{Q}\},\ \{(a,b) : a,b \in \mathbb{R}\},\ \mathcal{O}^1, \text{ or } \mathcal{C}^1$$

Here is just an example how to solve the problem. Let $b > a$. Since $(-\infty,b)\setminus(-\infty,a) = [a,b)$ we get that

$$\{[a,b) : a,b \in \mathbb{Q}\} \subset \sigma(\{(-\infty,c) : c \in \mathbb{Q}\})$$
$$\implies \mathcal{B} = \sigma(\{[a,b) : a,b \in \mathbb{Q}\}) \subset \sigma(\{(-\infty,c) : c \in \mathbb{Q}\}).$$

On the other hand we find that $(-\infty,a) = \bigcup_{k\in\mathbb{N}}[-k,a)$ proving that

$$\{(-\infty,a) : a \in \mathbb{Q}\} \subset \sigma(\{[c,d) : c,d \in \mathbb{Q}\}) = \mathcal{B}$$
$$\implies \sigma(\{(-\infty,a) : a \in \mathbb{Q}\}) \subset \mathcal{B}$$

and we get equality.

The other cases are similar.

Problem 3.9 Let $\mathbb{B} := \{B_r(x) : x \in \mathbb{R}^n,\ r > 0\}$ and let $\mathbb{B}' := \{B_r(x) : x \in \mathbb{Q}^n,\ r \in \mathbb{Q}^+\}$. Clearly,

$$\mathbb{B}' \subset \mathbb{B} \subset \mathcal{O}^n$$
$$\implies \sigma(\mathbb{B}') \subset \sigma(\mathbb{B}) \subset \sigma(\mathcal{O}^n) = \mathcal{B}(\mathbb{R}^n).$$

On the other hand, any open set $U \in \mathcal{O}^n$ can be represented by

$$U = \bigcup_{B\in\mathbb{B}',\, B\subset U} B. \tag{$*$}$$

Indeed, $U \supset \bigcup_{B\in\mathbb{B}',\, B\subset U} B$ follows by the very definition of the union. Conversely, if $x \in U$ we use the fact that U is open, i.e. there is some $B_\epsilon(x) \subset U$. Without loss of generality we can assume that ϵ is rational, otherwise we replace it by some smaller rational ϵ. Since $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$ we can find some $q \in \mathbb{Q}^n$ with $|x-q| < \epsilon/3$ and it is clear that $B_{\epsilon/3}(q) \subset B_\epsilon(x) \subset U$. This shows that $U \subset \bigcup_{B\in\mathbb{B}',\, B\subset U} B$.

Since $\#\mathbb{B}' = \#(\mathbb{Q}^n \times \mathbb{Q}) = \#\mathbb{N}$, formula $(*)$ entails that

$$\mathcal{O}^n \subset \sigma(\mathbb{B}') \implies \sigma(\mathcal{O}^n) = \sigma(\mathbb{B})$$

and we are done.

Problem 3.10 (i) $\mathcal{O}_1$: We have $\varnothing = \varnothing \cap A \in \mathcal{O}_A$, $A = X \cap A \in \mathcal{O}_A$.

$\mathcal{O}_1$: Let $U' = U \cap A \in \mathcal{O}_A$, $V' = V \cap A \in \mathcal{O}_A$ with $U, V \in \mathcal{O}$. Then $U' \cap V' = (U \cap V) \cap A \in \mathcal{O}_A$ since $U \cap V \in \mathcal{O}$.

$\mathcal{O}_2$: Let $U_i' = U_i \cap A \in \mathcal{O}_A$ with $U_i \in \mathcal{O}$. Then $\bigcup_i U_i' = (\bigcup_i U_i) \cap A \in \mathcal{O}_A$ since $\bigcup_i U_i \in \mathcal{O}$.

(ii) We use for a set A and a family $\mathcal{F} \subset \mathcal{P}(X)$ the shorthand $A \cap \mathcal{F} := \{A \cap F : F \in \mathcal{F}\}$. Clearly, $A \cap \mathcal{O} \subset A \cap \sigma(\mathcal{O}) = A \cap \mathcal{B}(X)$. Since the latter is a σ-algebra, we have

$$\sigma(A \cap \mathcal{O}) \subset A \cap \mathcal{B}(X) \quad \text{i.e.} \quad \mathcal{B}(A) \subset A \cap \mathcal{B}(X).$$

For the converse inclusion we define the family

$$\Sigma := \{B \subset X : A \cap B \in \sigma(A \cap \mathcal{O})\}.$$

It is not hard to see that Σ is a σ-algebra and that $\mathcal{O} \subset \Sigma$. Thus $\mathcal{B}(X) = \sigma(\mathcal{O}) \subset \Sigma$ which means that

$$A \cap \mathcal{B}(X) \subset \sigma(A \cap \mathcal{O}).$$

Notice that this argument does not really need that $A \in \mathcal{B}(X)$. If, however, $A \in \mathcal{B}(X)$ we have in addition to $A \cap \mathcal{B}(X) = \mathcal{B}(A)$ that

$$\mathcal{B}(A) = \{B \subset A : B \in \mathcal{B}(X)\}$$

Problem 3.11 (i) As in the proof of Theorem 3.4 we set

$$\mathfrak{m}(\mathcal{E}) := \bigcap_{\substack{\mathcal{M} \text{ monotone class} \\ \mathcal{M} \supset \mathcal{E}}} \mathcal{M}. \tag{$*$}$$

Since the intersection $\mathcal{M} = \bigcap_{i \in I} \mathcal{M}_i$ of arbitrarily many monotone classes $\mathcal{M}_i$, $i \in I$, is again a monotone class [indeed: if $(A_j)_{j\in\mathbb{N}} \subset \mathcal{M}$, then $(A_j)_{j\in\mathbb{N}}$ is in every $\mathcal{M}_i$ and so are $\bigcup_j A_j$, $\bigcap_j A_j$; thus $\bigcup_j A_j, \bigcap_j A_j \in \mathcal{M}$] and $(*)$ is evidently the smallest monotone class containing some given family $\mathcal{E}$.

(ii) Since $\mathcal{E}$ is stable under complementation and contains the empty set we know that $X \in \mathcal{E}$. Thus, $\varnothing \in \Sigma$ and, by the very definition, Σ is stable under taking complements of its elements. If $(S_j)_{j\in\mathbb{N}} \subset \Sigma$, then $(S_j^c)_{j\in\mathbb{N}} \subset \sigma$ and

$$\bigcup_j S_j \in \mathfrak{m}(\mathcal{E}), \qquad \Big(\bigcup_j S_j\Big)^c = \bigcap_j S_j^c \in \mathfrak{m}(\mathcal{E})$$

which means that $\bigcup_j S_j \in \Sigma$.

(iii) $\mathcal{E} \subset \Sigma$: if $E \in \mathcal{E}$, then $E \in \mathfrak{m}(\mathcal{E})$. Moreover, as $\mathcal{E}$ is stable under complementation, $E^c \in \mathfrak{m}(\mathcal{E})$ for all $E \in \mathcal{E}$, i.e. $\mathcal{E} \subset \Sigma$.

$\Sigma \subset \mathfrak{m}(\mathcal{E})$: obvious from the definition of Σ.

$\mathfrak{m}(\mathcal{E}) \subset \sigma(\mathcal{E})$: every σ-algebra is also a monotone class and the inclusion follows from the minimality of $\mathfrak{m}(\mathcal{E})$.

Finally apply the σ-hull to the chain $\mathcal{E} \subset \Sigma \subset \mathfrak{m}(\mathcal{E}) \subset \sigma(\mathcal{E})$ and conclude that $m(\mathcal{E}) \subset \sigma(\mathcal{E})$.

Problem 3.12 (i) Since $\mathcal{M}$ is a monotone class, this follows from Problem 3.11.

(ii) Let $F \subset \mathbb{R}^n$ be any closed set. Then $U_n := F + B_{1/n}(0) := \{x + y : x \in F,\ y \in B_{1/n}(0)\}$ is an open set and $\bigcap_{n\in\mathbb{N}} U_n = F$. Indeed,

$$U_n = \bigcup_{x\in F} B_{1/n}(x) = \left\{z \in \mathbb{R}^n : |x-z| < \tfrac{1}{n} \text{ for some } x \in F\right\}$$

which shows that U_n is open, $F \subset U_n$ and $F \subset \bigcap_n U_n$. On the other hand, if $z \in U_n$ for all $n \in \mathbb{N}$, then there is a sequence of points $x_n \in F$ with the property $|z - x_n| < \frac{1}{n} \xrightarrow{n\to\infty} 0$. Since F is closed, $z = \lim_n x_n \in F$ and we get $F = \bigcap_n U_n$.

Since $\mathcal{M}$ is closed under countable intersections, $F \in \mathcal{M}$ for any closed set F.

(iii) Identical to Problem 3.11(ii).

(iv) Use Problem 3.11(iv).

4 Measures. Solutions to Problems 4.1–4.15

Problem 4.1 (i) We have to show that for a measure μ and finitely many, pairwise disjoint sets $A_1, A_2, \dots, A_N \in \mathcal{A}$ we have

$$\mu(A_1 \uplus A_2 \uplus \dots \uplus A_N) = \mu(A_1) + \mu(A_2) + \dots + \mu(A_N).$$

We use induction in $N \in \mathbb{N}$. The hypothesis is clear, for the start ($N = 2$) see Proposition 4.3(i). Induction step: take $N + 1$ disjoint sets $A_1, \dots, A_{N+1} \in \mathcal{A}$, set $B := A_1 \uplus \dots \uplus A_N \in \mathcal{A}$ and use the induction start and the hypothesis to conclude

$$\begin{aligned}\mu(A_1 \uplus \dots \uplus A_N \uplus A_{N+1}) &= \mu(B \uplus A_{N+1}) \\ &= \mu(B) + \mu(A_{N+1}) \\ &= \mu(A_1) + \dots + \mu(A_N) + \mu(A_{N+1}).\end{aligned}$$

(iv) To get an idea what is going on we consider first the case of three sets A, B, C. Applying the formula for strong additivity thrice we get

$$\begin{aligned}\mu(A \cup B \cup C) &= \mu(A \cup (B \cup C)) \\ &= \mu(A) + \mu(B \cup C) - \mu(\underbrace{A \cap (B \cup C)}_{=(A\cap B)\cup(A\cap C)}) \\ &= \mu(A) + \mu(B) + \mu(C) - \mu(B \cap C) - \mu(A \cap B) \\ &\quad - \mu(A \cap C) + \mu(A \cap B \cap C).\end{aligned}$$

As an educated guess it seems reasonable to suggest that

$$\mu(A_1 \cup \dots \cup A_n) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{\substack{\sigma \subset \{1,\dots,n\} \\ \#\sigma = k}} \mu\Big(\bigcap_{j \in \sigma} A_j\Big).$$

We prove this formula by induction. The induction start is just the formula from Proposition 4.3(iv), the hypothesis is given above. For the induction step we observe that

$$\begin{aligned}\sum_{\substack{\sigma \subset \{1,\dots,n+1\} \\ \#\sigma = k}} &= \sum_{\substack{\sigma \subset \{1,\dots,n,n+1\} \\ \#\sigma = k,\ n+1 \notin \sigma}} + \sum_{\substack{\sigma \subset \{1,\dots,n,n+1\} \\ \#\sigma = k,\ n+1 \in \sigma}} \\ &= \sum_{\substack{\sigma \subset \{1,\dots,n\} \\ \#\sigma = k}} + \sum_{\substack{\sigma' \subset \{1,\dots,n\} \\ \#\sigma' = k-1,\ \sigma := \sigma' \cup \{n+1\}}}\end{aligned} \tag{$*$}$$

Having this in mind we get for $B := A_1 \cup \ldots \cup A_n$ and A_{n+1} using strong additivity and the induction hypothesis (for $A_1, \ldots, A_n$ resp. $A_1 \cap A_{n+1}, \ldots, A_n \cap A_{n+1}$)

$$\begin{aligned}\mu(B \cup A_{n+1}) &= \mu(B) + \mu(A_{n+1}) - \mu(B \cap A_{n+1}) \\ &= \mu(B) + \mu(A_{n+1}) - \mu\Big(\bigcup_{j=1}^{n}(A_j \cap A_{n+1})\Big) \\ &= \sum_{k=1}^{n}(-1)^{k+1} \sum_{\substack{\sigma\subset\{1,\ldots,n\}\\ \#\sigma=k}} \mu\Big(\bigcap_{j\in\sigma} A_j\Big) + \mu(A_{n+1}) \\ &\quad + \sum_{k=1}^{n}(-1)^{k+1} \sum_{\substack{\sigma\subset\{1,\ldots,n\}\\ \#\sigma=k}} \mu\Big(A_{n+1} \bigcap_{j\in\sigma} A_j\Big).\end{aligned}$$

Because of $(*)$ the last line coincides with

$$\sum_{k=1}^{n+1}(-1)^{k+1} \sum_{\substack{\sigma\subset\{1,\ldots,n,n+1\}\\ \#\sigma=k}} \mu\Big(\bigcap_{j\in\sigma} A_j\Big)$$

and the induction is complete.

(v) We have to show that for a measure μ and finitely many sets $B_1, B_2, \ldots, B_N \in \mathcal{A}$ we have

$$\mu(B_1 \cup B_2 \cup \ldots \cup B_N) \leqslant \mu(B_1) + \mu(B_2) + \ldots + \mu(B_N).$$

We use induction in $N \in \mathbb{N}$. The hypothesis is clear, for the start ($N = 2$) see Proposition 4.3(v). Induction step: take $N+1$ sets $B_1, \ldots, B_{N+1} \in \mathcal{A}$, set $C := B_1 \cup \ldots \cup B_N \in \mathcal{A}$ and use the induction start and the hypothesis to conclude

$$\begin{aligned}\mu(B_1 \cup \ldots \cup B_N \cup B_{N+1}) &= \mu(C \cup B_{N+1}) \\ &\leqslant \mu(C) + \mu(B_{N+1}) \\ &\leqslant \mu(B_1) + \ldots + \mu(B_N) + \mu(B_{N+1}).\end{aligned}$$

Problem 4.2 (i) The Dirac measure is defined on an arbitrary measurable space $(X, \mathcal{A})$ by

$$\delta_x(A) := \begin{cases} 0, & \text{if } x \notin A \\ 1, & \text{if } x \in A \end{cases}, \text{ where } A \in \mathcal{A} \text{ and } x \in X \text{ is a fixed point.}$$

<u>(M_1)</u> Since $\varnothing$ contains no points, $x \notin \varnothing$ and so $\delta_x(\varnothing) = 0$.

<u>(M_2)</u> Let $(A_j)_{j\in\mathbb{N}} \subset \mathcal{A}$ a sequence of pairwise disjoint measurable sets. If $x \in \bigcup_{j\in\mathbb{N}} A_j$, there is exactly one j_0 with $x \in A_{j_0}$, hence

$$\begin{aligned}\delta_x\Bigg(\bigcup_{j\in\mathbb{N}} A_j\Bigg) &= 1 = 1 + 0 + 0 + \ldots \\ &= \delta_x(A_{j_0}) + \sum_{j\neq j_0} \delta_x(A_j) \\ &= \sum_{j\in\mathbb{N}} \delta_x(A_j).\end{aligned}$$

If $x \notin \bigcup_{j\in\mathbb{N}} A_j$, then $x \notin A_j$ for every $j \in \mathbb{N}$, hence

$$\delta_x\left(\bigcup_{j\in\mathbb{N}} A_j\right) = 0 = 0 + 0 + 0 + \ldots = \sum_{j\in\mathbb{N}} \delta_x(A_j).$$

(ii) The measure γ is defined on $(\mathbb{R}, \mathcal{A})$ by $\gamma(A) := \begin{cases} 0, \text{ if } \#A \leqslant \#\mathbb{N} \\ 1, \text{ if } \#A^c \leqslant \#\mathbb{N} \end{cases}$ where $\mathcal{A} := \{A \subset \mathbb{R} : \#A \leqslant \#\mathbb{N} \text{ or } \#A^c \leqslant \#\mathbb{N}\}$. (Note that $\#A \leqslant \#\mathbb{N}$ if, and only if, $\#A^c = \#\mathbb{R}\setminus A > \#\mathbb{N}$.)

<u>(M_1)</u> Since $\varnothing$ contains no elements, it is certainly countable and so $\gamma(\varnothing) = 0$.

<u>(M_2)</u> Let $(A_j)_{j\in\mathbb{N}}$ be pairwise disjoint $\mathcal{A}$-sets. If all of them are countable, then $A := \bigcup_{j\in\mathbb{N}}$ is countable and we get

$$\gamma\left(\bigcup_{j\in\mathbb{N}} A_j\right) = \gamma(A) = 0 = \sum_{j\in\mathbb{N}} \gamma(A_j).$$

If at least one A_j is not countable, say for $j = j_0$, then $A \supset A_{j_0}$ is not countable and therefore $\gamma(A) = \gamma(A_{j_0}) = 1$. Assume we could find some other $j_1 \neq j_0$ such that A_{j_0}, A_{j_1} are not countable. Since $A_{j_0}, A_{j_1} \in \mathcal{A}$ we know that their complements $A_{j_0}^c, A_{j_1}^c$ are countable, hence $A_{j_0}^c \cup A_{j_1}^c$ is countable and, at the same time, $\in \mathcal{A}$. Because of this, $(A_{j_0}^c \cup A_{j_1}^c)^c = A_{j_0} \cap A_{j_1} = \varnothing$ cannot be countable, which is absurd! Therefore there is at most one index $j_0 \in \mathbb{N}$ such that A_{j_0} is uncountable and we get then

$$\gamma\left(\bigcup_{j\in\mathbb{N}} A_j\right) = \gamma(A) = 1 = 1 + 0 + 0 + \ldots = \gamma(A_{j_0}) + \sum_{j\neq j_0} \gamma(A_j).$$

(iii) We have an arbitrary measurable space $(X, \mathcal{A})$ and the measure

$$|A| = \begin{cases} \#A, & \text{if } A \text{ is finite} \\ \infty, & \text{else} \end{cases}.$$

<u>(M_1)</u> Since $\varnothing$ contains no elements, $\#\varnothing = 0$ and $|\varnothing| = 0$.

<u>(M_2)</u> Let $(A_j)_{j\in\mathbb{N}}$ be a sequence of pairwise disjoint sets in $\mathcal{A}$. Case 1: All A_j are finite and only finitely many, say the first k, are non-empty, then $A = \bigcup_{j\in\mathbb{N}} A_j$ is effectively a finite union of k finite sets and it is clear that

$$|A| = |A_1| + \ldots + |A_k| + |\varnothing| + |\varnothing| + \ldots = \sum_{j\in\mathbb{N}} |A_j|.$$

Case 2: All A_j are finite and infinitely many are non-void. Then their union $A = \bigcup_{j\in\mathbb{N}} A_j$ is an infinite set and we get

$$|A| = \infty = \sum_{j\in\mathbb{N}} |A_j|.$$

Case 3: At least one A_j is infinite, and so is then the union $A = \bigcup_{j\in\mathbb{N}} A_j$. Thus,

$$|A| = \infty = \sum_{j\in\mathbb{N}} |A_j|.$$

(iv) On a countable set $\Omega = \{\omega_1, \omega_2, \dots\}$ we define for a sequence $(p_j)_{j\in\mathbb{N}} \subset [0,1]$ with $\sum_{j\in\mathbb{N}} p_j = 1$ the set-function

$$P(A) = \sum_{j\,:\,\omega_j\in A} p_j = \sum_{j\in\mathbb{N}} p_j\,\delta_{\omega_j}(A), \qquad A\subset\Omega.$$

$\underline{(M_1)}$ $P(\emptyset) = 0$ is obvious.

$\underline{(M_2)}$ Let $(A_k)_{k\in\mathbb{N}}$ be pairwise disjoint subsets of Ω. Then

$$\begin{aligned}
\sum_{k\in\mathbb{N}} P(A_k) &= \sum_{k\in\mathbb{N}}\sum_{j\in\mathbb{N}} p_j\,\delta_{\omega_j}(A_k)\\
&= \sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}} p_j\,\delta_{\omega_j}(A_k)\\
&= \sum_{j\in\mathbb{N}} p_j\Big(\sum_{k\in\mathbb{N}}\delta_{\omega_j}(A_k)\Big)\\
&= \sum_{j\in\mathbb{N}} p_j\delta_{\omega_j}\big(\textstyle\bigcup_k A_k\big)\\
&= P\big(\textstyle\bigcup_k A_k\big).
\end{aligned}$$

The change in the order of summation needs justification; one possibility is the argument used in the solution of Problem 4.6(ii). (Note that the reordering theorem for absolutely convergent series is not immediately applicable since we deal with a double series!)

(v) This is obvious.

Problem 4.3 • On $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ the function γ is not be a measure, since we can take the sets $A = (1,\infty)$, $B = (-\infty,-1)$ which are disjoint, not countable and both have non-countable complements. Hence, $\gamma(A) = \gamma(B) = 1$. On the other hand, $A\cup B$ is non-countable and has non-countable complement, $[-1,1]$. So, $\gamma(A\cup B) = 1$. This contradicts the additivity: $\gamma(A\cup B) = 1 \neq 2 = \gamma(A)+\gamma(B)$. *Notice that the choice of the σ-algebra $\mathcal{A}$ avoids exactly this situation. $\mathcal{B}$ is the wrong σ-algebra for γ.*

• On $\mathbb{Q}$ (and, actually, any possible σ-algebra thereon) the problem is totally different: if A is countable, then $A^c = \mathbb{Q}\setminus A$ is also countable and vice versa. This means that $\gamma(A)$ is, according to the definition, both 1 and 0 which is, of course, impossible. This is to say: γ is not well-defined. *γ makes only sense on a non-countable set X.*

Problem 4.4 (i) If $\mathcal{A} = \{\emptyset, \mathbb{R}\}$, then μ is a measure.

But as soon as $\mathcal{A}$ contains one set A which is trivial (i.e. either $\emptyset$ or X), we have actually $A^c \in \mathcal{A}$ which is also non-trivial. Thus,

$$1 = \mu(X) = \mu(A\cup A^c) \neq \mu(A)+\mu(A^c) = 1+1 = 2$$

and μ cannot be a measure.

(ii) If we equip $\mathbb{R}$ with a σ-algebra which contains sets such that both A and A^c can be infinite (the Borel σ-algebra would be such an example: $A = (-\infty, 0) \implies A^c = [0, \infty)$), then ν is not well-defined. The only type of sets where ν is well-defined is, thus,

$$\mathcal{A} := \{A \subset \mathbb{R} : \#A < \infty \text{ or } \#A^c < \infty\}.$$

But this is no σ-algebra as the following example shows: $A_j := \{j\} \in \mathcal{A}$, $j \in \mathbb{N}$, are pairwise disjoint sets but $\bigcup_{j\in\mathbb{N}} A_j = \mathbb{N}$ is not finite and its complement is $\mathbb{R} \setminus \mathbb{N}$ not finite either! Thus, $\mathbb{N} \notin \mathcal{A}$, showing that $\mathcal{A}$ cannot be a σ-algebra. We conclude that ν can never be a measure if the σ-algebra contains infinitely many sets. If we are happy with finitely many sets only, then here is an example that makes ν into a measure $\mathcal{A} = \{\emptyset, \{69\}, \mathbb{R} \setminus \{69\}, \mathbb{R}\}$ and similar families are possible, but the point is that they all contain only finitely many members.

Problem 4.5 Denote by λ one-dimensional Lebesgue measure and consider the Borel sets $B_k := (k, \infty)$. Clearly $\bigcap_k B_k = \emptyset$, $k \in \mathbb{N}$, so that $B_k \downarrow \emptyset$. On the other hand,

$$\lambda(B_k) = \infty \implies \inf_k \lambda(B_k) = \infty \neq 0 = \lambda(\emptyset)$$

which shows that the finiteness condition in Theorem 4.4 (iii′) and (iii″) is essential.

Problem 4.6 (i) Clearly, $\rho := a\mu + b\nu : \mathcal{A} \to [0, \infty]$ (since $a, b \geqslant 0$!). We check $(M_1), (M_2)$.

$\underline{(M_1)}$ Clearly, $\rho(\emptyset) = a\mu(\emptyset) + b\nu(\emptyset) = a \cdot 0 + b \cdot 0 = 0$.

$\underline{(M_2)}$ Let $(A_j)_{j\in\mathbb{N}} \subset \mathcal{A}$ be mutually disjoint sets. Then we can use the σ-additivity of μ, ν to get

$$\begin{aligned}
\rho\Big(\bigcup_{j\in\mathbb{N}} A_j\Big) &= a\mu\Big(\bigcup_{j\in\mathbb{N}} A_j\Big) + b\nu\Big(\bigcup_{j\in\mathbb{N}} A_j\Big) \\
&= a \sum_{j\in\mathbb{N}} \mu(A_j) + b \sum_{j\in\mathbb{N}} \nu(A_j) \\
&= \sum_{j\in\mathbb{N}} \big(a\mu(A_j) + b\mu(A_j)\big) \\
&= \sum_{j\in\mathbb{N}} \rho(A_j).
\end{aligned}$$

Since all quantities involved are positive and since we allow the value $+\infty$ to be attained, there are no convergence problems.

(ii) Since all α_j are positive, the sum $\sum_{j\in\mathbb{N}} \alpha_j \mu_j(A)$ is a sum of positive quantities and, allowing the value $+\infty$ to be attained, there is no convergence problem. Thus, $\mu : \mathcal{A} \to [0, \infty]$ is well-defined. Before we check $(M_1), (M_2)$ we prove the following

Lemma. *Let β_{ij}, $i, j \in \mathbb{N}$, be real numbers. Then*

$$\sup_{i\in\mathbb{N}} \sup_{j\in\mathbb{N}} \beta_{ij} = \sup_{j\in\mathbb{N}} \sup_{i\in\mathbb{N}} \beta_{ij}.$$

Proof. Observe that we have $\beta_{mn} \leqslant \sup_{j\in\mathbb{N}} \sup_{i\in\mathbb{N}} \beta_{ij}$ for all $m, n \in \mathbb{N}$. The right-hand side is independent of m and n and we may take the *sup* over all n

$$\sup_{n\in\mathbb{N}} \beta_{mn} \leqslant \sup_{j\in\mathbb{N}} \sup_{i\in\mathbb{N}} \beta_{ij} \qquad \forall m \in \mathbb{N}$$

and then, with the same argument, take the sup over all m

$$\sup_{m\in\mathbb{N}} \sup_{n\in\mathbb{N}} \beta_{mn} \leqslant \sup_{j\in\mathbb{N}} \sup_{i\in\mathbb{N}} \beta_{ij} \qquad \forall m \in \mathbb{N}.$$

The opposite inequality, '$\geqslant$', follows from the same argument with i and j interchanged. ∎

<u>(M_1)</u> We have $\mu(\emptyset) = \sum_{j\in\mathbb{N}} \alpha_j \mu_j(\emptyset) = \sum_{j\in\mathbb{N}} \alpha_j \cdot 0 = 0$.

<u>(M_2)</u> Take pairwise disjoint sets $(A_i)_{i\in\mathbb{N}} \subset \mathcal{A}$. Then we can use the σ-additivity of each of the μ_j's to get

$$\begin{aligned}
\mu\Big(\bigcup_{i\in\mathbb{N}} A_i\Big) &= \sum_{j\in\mathbb{N}} \alpha_j \mu_j \Big(\bigcup_{i\in\mathbb{N}} A_i\Big) \\
&= \lim_{N\to\infty} \sum_{j=1}^{N} \alpha_j \sum_{i\in\mathbb{N}} \mu_j(A_i) \\
&= \lim_{N\to\infty} \sum_{j=1}^{N} \alpha_j \lim_{M\to\infty} \sum_{i=1}^{M} \mu_j(A_i) \\
&= \lim_{N\to\infty} \lim_{M\to\infty} \sum_{j=1}^{N} \sum_{i=1}^{M} \alpha_j \mu_j(A_i) \\
&= \sup_{N\in\mathbb{N}} \sup_{M\in\mathbb{N}} \sum_{j=1}^{N} \sum_{i=1}^{M} \alpha_j \mu_j(A_i)
\end{aligned}$$

where we used that the limits are increasing limits, hence suprema. By our lemma:

$$\begin{aligned}
\mu\Big(\bigcup_{i\in\mathbb{N}} A_i\Big) &= \sup_{M\in\mathbb{N}} \sup_{N\in\mathbb{N}} \sum_{i=1}^{M} \sum_{j=1}^{N} \alpha_j \mu_j(A_i) \\
&= \lim_{M\to\infty} \lim_{N\to\infty} \sum_{i=1}^{M} \sum_{j=1}^{N} \alpha_j \mu_j(A_i) \\
&= \lim_{M\to\infty} \sum_{i=1}^{M} \sum_{j\in\mathbb{N}} \alpha_j \mu_j(A_i) \\
&= \lim_{M\to\infty} \sum_{i=1}^{M} \mu(A_i) \\
&= \sum_{i\in\mathbb{N}} \mu(A_i).
\end{aligned}$$

Problem 4.7 Set $\nu(A) := \mu(A \cap F)$. We know, by assumption, that μ is a measure on $(X, \mathcal{A})$. We have to show that ν is a measure on $(X, \mathcal{A})$. Since $F \in \mathcal{A}$, we have $F \cap A \in \mathcal{A}$ for all $A \in \mathcal{A}$, so ν is well-defined. Moreover, it is clear that $\nu(A) \in [0, \infty]$. Thus, we only have to check

$\underline{(M_1)}$ $\nu(\varnothing) = \mu(\varnothing \cap F) = \mu(\varnothing) = 0$.

$\underline{(M_2)}$ Let $(A_j)_{j\in\mathbb{N}} \subset \mathcal{A}$ be a sequence of pairwise disjoint sets. Then also $(A_j \cap F)_{j\in\mathbb{N}} \subset \mathcal{A}$ are pairwise disjoint and we can use the σ-additivity of μ to get

$$\begin{aligned}\nu\left(\bigcup_{j\in\mathbb{N}} A_j\right) = \mu\left(F \cap \bigcup_{j\in\mathbb{N}} A_j\right) &= \mu\left(\bigcup_{j\in\mathbb{N}} (F \cap A_j)\right)\\ &= \sum_{j\in\mathbb{N}} \mu(F \cap A_j)\\ &= \sum_{j\in\mathbb{N}} \nu(A_j).\end{aligned}$$

Problem 4.8 Since P is a probability measure, $P(A_j^c) = 1 - P(A_j) = 0$. By σ-subadditivity,

$$P\left(\bigcup_{j\in\mathbb{N}} A_j^c\right) \leqslant \sum_{j\in\mathbb{N}} P(A_j^c), = 0$$

and we conclude that

$$P\left(\bigcap_{j\in\mathbb{N}} A_j\right) = 1 - P\left(\left[\bigcap_{j\in\mathbb{N}} A_j\right]^c\right) = 1 - P\left(\bigcup_{j\in\mathbb{N}} A_j^c\right) = 1 - 0 = 0.$$

Problem 4.9 Note that

$$\bigcup_j A_j \setminus \bigcup_k B_k = \bigcup_j \Bigg(A_j \setminus \underbrace{\bigcup_k B_k}_{\supset B_j\ \forall j}\Bigg) \subset \bigcup_j \big(A_j \setminus B_j\big)$$

Since $\bigcup_j B_j \subset \bigcup_j A_j$ we get from σ-subadditivity

$$\begin{aligned}\mu\left(\bigcup_j A_j\right) - \mu\left(\bigcup_j B_j\right) &= \mu\left(\bigcup_j A_j \setminus \bigcup_k B_k\right)\\ &\leqslant \mu\left(\bigcup_j \big(A_j \setminus B_j\big)\right)\\ &\leqslant \sum_j \mu(A_j \setminus B_j).\end{aligned}$$

Problem 4.10 (i) We have $\varnothing \in \mathcal{A}$ and $\mu(\varnothing) = 0$, thus $\varnothing \in \mathcal{N}_\mu$.

(ii) Since $M \in \mathcal{A}$ (this is essential in order to apply μ to M!) we can use the monotonicity of measures to get $0 \leqslant \mu(M) \leqslant \mu(N) = 0$, i.e. $\mu(M) = 0$ and $M \in \mathcal{N}_\mu$ follows.

(iii) Since all $N_j \in \mathcal{A}$, we get $N := \bigcup_{j\in\mathbb{N}} N_j \in \mathcal{A}$. By the σ-subadditivity of a measure we find

$$0 \leqslant \mu(N) = \mu\left(\bigcup_{j\in\mathbb{N}} N_j\right) \leqslant \sum_{j\in\mathbb{N}} \mu(N_j) = 0,$$

hence $\mu(N) = 0$ and so $N \in \mathcal{N}_\mu$.

Problem 4.11 (i) The one-dimensional Borel sets $\mathcal{B} := \mathcal{B}^1$ are defined as the smallest σ-algebra containing the open sets. Pick $x \in \mathbb{R}$ and observe that the open intervals $(x - \frac{1}{k}, x + \frac{1}{k})$, $k \in \mathbb{N}$, are all open sets and therefore $(x - \frac{1}{k}, x + \frac{1}{k}) \in \mathcal{B}$. Since a σ-algebra is stable under countable intersections we get $\{x\} = \bigcap_{k\in\mathbb{N}}(x - \frac{1}{k}, x + \frac{1}{k}) \in \mathcal{B}$.

Using the monotonicity of measures and the definition of Lebesgue measure we find

$$0 \leqslant \lambda(\{x\}) \leqslant \lambda((x - \tfrac{1}{k}, x + \tfrac{1}{k})) = (x + \tfrac{1}{k}) - (x - \tfrac{1}{k}) = \tfrac{2}{k} \xrightarrow{k\to\infty} 0.$$

[Following the hint leads to a similar proof with $[x - \frac{1}{k}, x + \frac{1}{k})$ instead of $(x - \frac{1}{k}, x + \frac{1}{k})$.]

(ii) a) Since $\mathbb{Q}$ is countable, we find an enumeration $\{q_1, q_2, q_3, \ldots\}$ and we get trivially $\mathbb{Q} = \bigcup_{j\in\mathbb{N}}\{q_j\}$ which is a disjoint union. (This shows, by the way, that $\mathbb{Q} \in \mathcal{B}$ as $\{q_j\} \in \mathcal{B}$.) Therefore, using part (i) of the problem and the σ-additivity of measures,

$$\lambda(\mathbb{Q}) = \lambda\Bigg(\bigcup_{j\in\mathbb{N}}\{q_j\}\Bigg) = \sum_{j\in\mathbb{N}}\lambda(\{q_j\}) = \sum_{j\in\mathbb{N}} 0 = 0.$$

b) Take again an enumeration $\mathbb{Q} = \{q_1, q_2, q_3, \ldots\}$, fix $\epsilon > 0$ and define $C(\epsilon)$ as stated in the problem. Then we have $C(\epsilon) \in \mathcal{B}$ and $\mathbb{Q} \subset C(\epsilon)$. Using the monotonicity and σ-subadditivity of λ we get

$$\begin{aligned}
0 \leqslant \lambda(\mathbb{Q}) &\leqslant \lambda\big(C(\epsilon)\big) \\
&= \lambda\Bigg(\bigcup_{k\in\mathbb{N}}[q_k - \epsilon 2^{-k}, q_k + \epsilon 2^{-k})\Bigg) \\
&\leqslant \sum_{k\in\mathbb{N}} \lambda\big([q_k - \epsilon 2^{-k}, q_k + \epsilon 2^{-k})\big) \\
&= \sum_{k\in\mathbb{N}} 2 \cdot \epsilon \cdot 2^{-k} \\
&= 2\epsilon \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 2\epsilon.
\end{aligned}$$

As $\epsilon > 0$ was arbitrary, we can make $\epsilon \to 0$ and the claim follows.

(iii) Since $\bigcup_{0\leqslant x\leqslant 1}\{x\}$ is a disjoint union, only the countability assumption is violated. Let's see what happens if we could use 'σ-additivity' for such non-countable unions:

$$0 = \sum_{0\leqslant x\leqslant 1} 0 = \sum_{0\leqslant x\leqslant 1} \lambda(\{x\}) = \lambda\Bigg(\bigcup_{0\leqslant x\leqslant 1}\{x\}\Bigg) = \lambda([0,1]) = 1$$

which is impossible.

Problem 4.12 Without loss of generality we may assume that $a \neq b$; set $\mu := \delta_a + \delta_b$. Then $\mu(B) = 0$ if, and only if, $a \notin B$ and $b \notin B$. Since $\{a\}, \{b\}$ and $\{a, b\}$ are Borel sets, all null sets of μ are given by

$$\mathcal{N}_\mu = \big\{B \setminus \{a, b\} \;:\; B \in \mathcal{B}(\mathbb{R})\big\}.$$

(This shows that, in some sense, null sets can be fairly large!).

Problem 4.13 Let us write $\mathfrak{N}$ for the family of all (proper and improper) subsets of μ null sets. We note that sets in $\mathfrak{N}$ can be measurable (that is: $N \in \mathcal{A}$) but need not be measurable.

(i) Since $\varnothing \in \mathfrak{N}$, we find that $A = A \cup \varnothing \in \mathcal{A}^*$ for every $A \in \mathcal{A}$; thus, $\mathcal{A} \subset \mathcal{A}^*$. Let us check that $\mathcal{A}^*$ is a σ-algebra.

(Σ_1) Since $\varnothing \in \mathcal{A} \subset \mathcal{A}^*$, we have $\varnothing \in \mathcal{A}^*$.

(Σ_2) Let $A^* \in \mathcal{A}^*$. Then $A^* = A \cup N$ for $A \in \mathcal{A}$ and $N \in \mathfrak{N}$. By definition, $N \subset M \in \mathcal{A}$ where $\mu(M) = 0$. Now

$$\begin{aligned} A^{*c} = (A \cup N)^c &= A^c \cap N^c \\ &= A^c \cap N^c \cap (M^c \cup M) \\ &= (A^c \cap N^c \cap M^c) \cup (A^c \cap N^c \cap M) \\ &= (A^c \cap M^c) \cup (A^c \cap N^c \cap M) \end{aligned}$$

where we used that $N \subset M$, hence $M^c \subset N^c$, hence $M^c \cap N^c = M^c$. But now we see that $A^c \cap M^c \in \mathcal{A}$ and $A^c \cap N^c \cap M \in \mathfrak{N}$ since $A^c \cap N^c \cap M \subset M$ and $M \in \mathcal{A}$ is a μ null set: $\mu(M) = 0$.

(Σ_3) Let $(A_j^*)_{j\in\mathbb{N}}$ be a sequence of $\mathcal{A}^*$-sets. From its very definition we know that each $A_j^* = A_j \cup N_j$ for some (not necessarily unique!) $A_j \in \mathcal{A}$ and $N_j \in \mathfrak{N}$. So,

$$\bigcup_{j\in\mathbb{N}} A_j^* = \bigcup_{j\in\mathbb{N}} (A_j \cup N_j) = \left(\bigcup_{j\in\mathbb{N}} A_j\right) \cup \left(\bigcup_{j\in\mathbb{N}} N_j\right) =: A \cup N.$$

Since $\mathcal{A}$ is a σ-algebra, $A \in \mathcal{A}$. All we have to show is that N_j is in $\mathfrak{N}$. Since each N_j is a subset of a (measurable!) null set, say, $M_j \in \mathcal{A}$, we find that $N = \bigcup_{j\in\mathbb{N}} N_j \subset \bigcup_{j\in\mathbb{N}} M_j = M \in \mathcal{A}$ and all we have to show is that $\mu(M) = 0$. But this follows from σ-subadditivity,

$$0 \leqslant \mu(M) = \mu\left(\bigcup_{j\in\mathbb{N}} M_j\right) \leqslant \sum_{j\in\mathbb{N}} \mu(M_j) = 0.$$

Thus, $A \cup N \in \mathcal{A}^*$.

(ii) As already mentioned in part (i), $A^* \in \mathcal{A}^*$ could have more than one representation, e.g. $A \cup N = A^* = B \cup M$ with $A, B \in \mathcal{A}$ and $N, M \in \mathfrak{N}$. If we can show that $\mu(A) = \mu(B)$ then the definition of $\bar{\mu}$ is independent of the representation of A^*. Since M, N are not necessarily measurable but, by definition, subsets of (measurable) null sets $M', N' \in \mathcal{A}$ we find

$$\begin{aligned} A \subset A \cup N = B \cup M \subset B \cup M', \\ B \subset B \cup M = A \cup N \subset A \cup N' \end{aligned}$$

and since $A, B, B \cup M', A \cup N' \in \mathcal{A}$, we get from monotonicity and subadditivity of measures

$$\mu(A) \leqslant \mu(B \cup M') \leqslant \mu(B) + \mu(M') = \mu(B),$$

$$\mu(B) \leqslant \mu(A \cup N') \leqslant \mu(A) + \mu(N') = \mu(A)$$

which shows $\mu(A) = \mu(B)$.

(iii) We check (M_1) and (M_2)

(M_1) Since $\emptyset = \emptyset \cup \emptyset \in \mathcal{A}^*$, $\emptyset \in \mathcal{A}$, $\emptyset \in \mathfrak{N}$, we have $\bar\mu(\emptyset) = \mu(\emptyset) = 0$.

(M_2) Let $(A_j^*)_{j\in\mathbb{N}} \subset \mathcal{A}^*$ be a sequence of pairwise disjoint sets. Then $A_j^* = A_j \cup N_j$ for some $A_j \in \mathcal{A}$ and $N_j \in \mathfrak{N}$. These sets are also mutually disjoint, and with the arguments in (i) we see that $A^* = A \cup N$ where $A^* \in \mathcal{A}^*, A \in \mathcal{A}, N \in \mathfrak{N}$ stand for the unions of A_j^*, A_j and N_j, respectively. Since $\bar\mu$ does not depend on the special representation of $\mathcal{A}^*$-sets, we get

$$\begin{aligned}\bar\mu\Big(\bigcup_{j\in\mathbb{N}} A_j^*\Big) = \bar\mu(A^*) = \mu(A) &= \mu\Big(\bigcup_{j\in\mathbb{N}} A_j\Big)\\ &= \sum_{j\in\mathbb{N}} \mu(A_j)\\ &= \sum_{j\in\mathbb{N}} \bar\mu(A_j^*)\end{aligned}$$

showing that $\bar\mu$ is σ-additive.

(iv) Let M^* be a $\bar\mu$ null set, i.e. $M^* \in \mathcal{A}^*$ and $\bar\mu(M^*) = 0$. Take any $B \subset M^*$. We have to show that $B \in \mathcal{A}^*$ and $\bar\mu(B) = 0$. The latter is clear from the monotonicity of $\bar\mu$ once we have shown that $B \in \mathcal{A}^*$ which means, once we know that we may plug B into $\bar\mu$.

Now, $B \subset M^*$ and $M^* = M \cup N$ for some $M \in \mathcal{A}$ and $N \in \mathfrak{N}$. As $\bar\mu(M^*) = 0$ we also know that $\mu(M) = 0$. Moreover, we know from the definition of $\mathfrak{N}$ that $N \subset N'$ for some $N' \in \mathcal{A}$ with $\mu(N') = 0$. This entails

$$B \subset M^* = M \cup N \subset M \cup N' \in \mathcal{A}$$
$$\text{and}\quad \mu(M \cup N') \leqslant \mu(M) + \mu(N') = 0.$$

Hence $B \in \mathfrak{N}$ as well as $B = \emptyset \cup B \in \mathcal{A}^*$. In particular, $\bar\mu(B) = \mu(\emptyset) = 0$.

(v) Set $\mathcal{C} = \{A^* \subset X \;:\; \exists A, B \in \mathcal{A},\quad A \subset A^* A \subset B,\quad \mu(B \setminus A) = 0\}$. We have to show that $\mathcal{A}^* = \mathcal{C}$.

Take $A^* \in \mathcal{A}^*$. Then $A^* = A \cup N$ with $A \in \mathcal{A}$, $N \in \mathfrak{N}$ and choose $N' \in \mathcal{A}$, $N \subset N'$ and $\mu(N') = 0$. This shows that

$$A \subset A^* = A \cup N \subset A \cup N' =: B \in \mathcal{A}$$

and that $\mu(B \setminus A) = \mu((A \cup N') \setminus A) \leqslant \mu(N') = 0$. (Note that $(A \cup N') \setminus A = (A \cup N') \cap A^c = N' \cap A^c \subset N'$ and that equality need not hold!).

Conversely, take $A^* \in \mathcal{C}$. Then, by definition, $A \subset A^* \subset B$ with $A, B \in \mathcal{A}$ and $\mu(B \setminus A) = 0$. Therefore, $N := B \setminus A$ is a null set and we see that $A^* \setminus A \subset B \setminus A$, i.e. $A^* \setminus A \in \mathfrak{N}$. So, $A^* = A \cup (A^* \setminus A)$ where $A \in \mathcal{A}$ and $A^* \setminus A \in \mathfrak{N}$ showing that $A^* \in \mathcal{A}^*$.

Problem 4.14 (i) Since $\mathcal{B}$ is a σ-algebra, it is closed under countable (disjoint) unions of its elements, thus ν inherits the properties (M_1), (M_2) directly from μ.

(ii) Yes [yes], since the full space $X \in \mathcal{B}$ so that $\mu(X) = \nu(X)$ is finite [resp. = 1].

(iii) No, σ-finiteness is also a property of the σ-algebra. Take, for example, Lebesgue measure λ on the Borel sets (this is σ-finite) and consider the σ-algebra $\mathcal{C} := \{\emptyset, (-\infty, 0), [0, \infty), \mathbb{R}\}$. Then $\lambda\big|_{\mathcal{C}}$ is not σ-finite since there is no increasing sequence of $\mathcal{C}$-sets having finite measure.

Problem 4.15 By definition, μ is σ-finite if there is an *increasing* sequence $(B_j)_{j\in\mathbb{N}} \subset \mathcal{A}$ such that $B_j \uparrow X$ and $\mu(B_j) < \infty$. Clearly, $E_j := B_j$ satisfies the condition in the statement of the problem.

Conversely, let $(E_j)_{j\in\mathbb{N}}$ be as stated in the problem. Then $B_n := E_1 \cup \ldots \cup E_n$ is measurable, $B_n \uparrow X$ and, by subadditivity,

$$\mu(B_n) = \mu(E_1 \cup \ldots \cup E_n) \leqslant \sum_{j=1}^{n} \mu(E_j) < \infty.$$

Remark: A small change in the above argument allows to take pairwise disjoint sets E_j.

5 Uniqueness of measures. Solutions to Problems 5.1–5.10

Problem 5.1 Since $X \in \mathcal{D}$ and since complements are again in $\mathcal{D}$, we have $\emptyset = X^c \in \mathcal{D}$.

If $A, B \in \mathcal{D}$ are disjoint, we set $A_1 := A, A_2 := B, A_j := \emptyset\ \forall j \geqslant 3$. Then $(A_j)_{j\in\mathbb{N}} \subset \mathcal{D}$ is a sequence of pairwise disjoint sets, and by (Δ_3) we find that

$$A \cup B = \dot{\bigcup_{j\in\mathbb{N}}} A_j \in \mathcal{D}.$$

Since $(\Sigma_1) = (\Delta_3)$, $(\Sigma_2) = (\Delta_2)$ and since $(\Sigma_3) \implies (\Delta_3)$, it is clear that every σ-algebra is also a Dynkin system; that the converse is, in general, wrong is seen in Problem 5.2.

Problem 5.2 Consider (Δ_3) only, as the other two conditions coincide: $(\Sigma_j) = (\Delta_j)$, $j = 1, 2$. We show that (Σ_3) breaks down even for finite unions. If $A, B \in \mathcal{D}$ are disjoint, it is clear that A, B *and also* $A \dot\cup B$ contain an even number of elements. But if A, B have non-void intersection, and if this intersection contains an odd number of elements, then $A \cup B$ contains an odd number of elements. Here is a trivial example:

$$A = \{1, 2\} \in \mathcal{D}, \quad B = \{2, 3, 4, 5\} \in \mathcal{D},$$

whereas

$$A \cup B = \{1, 2, 3, 4, 5\} \notin \mathcal{D}.$$

This means that (Δ_3) holds, but (Σ_3) fails.

Problem 5.3 *Mind the* ***misprint:*** *$A \subset B$ must be assumed and is* ***missing*** *in the statement of the problem!* We verify the hint first. Using de Morgan's laws we get

$$R \setminus Q = R \setminus (R \cap Q) = R \cap (R \cap Q)^c = (R^c \cup (R \cap Q))^c = (R^c \dot\cup (R \cap Q))^c$$

where the last equality follows since $R^c \cap (R \cap Q) = \emptyset$.

Now we take $A, B \in \mathcal{D}$ such that $A \subset B$. In particular $A \cap B = A$. Taking this into account and setting $Q = A, R = B$ we get from the above relation

$$B \setminus A = \Big(\underbrace{\underbrace{B^c}_{\in\mathcal{D}} \dot\cup A}_{\in\mathcal{D}} \Big)^c \in \mathcal{D}$$

where we repeatedly use (Δ_2) and (Δ_2).

Problem 5.4 (i) Since the σ-algebra $\mathcal{A}$ is also a Dynkin system, it is enough to prove $\delta(\mathcal{D}) = \mathcal{D}$ for any Dynkin system $\mathcal{D}$. By definition, $\delta(\mathcal{D})$ is the smallest Dynkin system containing $\mathcal{D}$, thus $\mathcal{D} \subset \delta(\mathcal{D})$. On the other hand, $\mathcal{D}$ is itself a Dynkin system, thus, because of minimality, $\mathcal{D} \supset \delta(\mathcal{D})$.

(ii) Clearly, $\mathcal{G} \subset \mathcal{H} \subset \delta(\mathcal{H})$. Since $\delta(\mathcal{H})$ is a Dynkin system containing $\mathcal{G}$, the minimality of $\delta(\mathcal{G})$ implies that $\delta(\mathcal{G}) \subset \delta(\mathcal{H})$.

(iii) Since $\sigma(\mathcal{G})$ is a σ-algebra, it is also a Dynkin system. Since $\mathcal{G} \subset \sigma(\mathcal{G})$ we conclude (again, by minimality) that $\delta(\mathcal{G}) \subset \sigma(\mathcal{G})$.

Problem 5.5 Clearly, $\delta(\{A,B\}) \subset \sigma(\{A,B\})$ is always true.

By Theorem 5.5, $\delta(\{A,B\}) = \sigma(\{A,B\})$ if $\{A,B\}$ is $\cap$-stable, i.e. if $A = B$ or $A = B^c$ or if at least one of A, B is X or $\emptyset$.

Let us exclude these cases. If $A \cap B = \emptyset$, then

$$\delta(\{A,B\}) = \sigma(\{A,B\}) = \big\{\emptyset, A, A^c, B, B^c, A \cup B, A^c \cap B^c, X\big\}.$$

If $A \cap B \neq \emptyset$, then

$$\delta(\{A,B\}) = \big\{\emptyset, A, A^c, B, B^c, X\big\}$$

while $\sigma(\{A,B\})$ is much larger containing, for example, $A \cap B$.

Problem 5.6 We prove the hint first. Let $(G_j)_{j\in\mathbb{N}} \subset \mathcal{G}$ as stated in the problem, i.e. satisfying (1) and (2), and define the sets $F_N := G_1 \cup \ldots \cup G_N$. As $\mathcal{G} \subset \mathcal{A}$, it is clear that $F_N \in \mathcal{A}$ (but not necessarily in $\mathcal{G}$...). Moreover, it is clear that $F_N \uparrow X$.

We begin with a more general assertion: *For any finite union of $\mathcal{G}$-sets $A_1 \cup \ldots \cup A_N$ we have $\mu(A_1 \cup \ldots \cup A_N) = \nu(A_1 \cup \ldots \cup A_N)$.*

Proof. *Induction Hypothesis:* $\mu(A_1 \cup \ldots \cup A_N) = \nu(A_1 \cup \ldots \cup A_N)$ for some $N \in \mathbb{N}$ and any choice of $A_1, \ldots, A_N \in \mathcal{G}$.

Induction Start $(N = 1)$*:* is obvious.

Induction Step $N \rightsquigarrow N+1$*:* We have by the strong additivity of measures and the $\cap$-stability of $\mathcal{G}$ that

$$\begin{aligned}
&\mu\big(A_1 \cup \ldots \cup A_N \cup A_{N+1}\big)\\
&= \mu\big((A_1 \cup \ldots \cup A_N) \cup A_{N+1}\big)\\
&= \mu\big(A_1 \cup \ldots \cup A_N\big) + \mu(A_{N+1}) - \mu\big((A_1 \cup \ldots \cup A_N) \cap A_{N+1}\big)\\
&= \mu\big(A_1 \cup \ldots \cup A_N\big) + \mu(A_{N+1}) - \mu\big(\underbrace{(A_1 \cap A_{N+1})}_{\in\mathcal{G}} \cup \ldots \cup \underbrace{(A_N \cap A_{N+1})}_{\in\mathcal{G}}\big)\\
&= \nu\big(A_1 \cup \ldots \cup A_N\big) + \nu(A_{N+1}) - \nu\big((A_1 \cap A_{N+1}) \cup \ldots \cup (A_N \cap A_{N+1})\big)\\
&\vdots
\end{aligned}$$

$$= \nu\big(A_1 \cup \ldots \cup A_N \cup A_{N+1}\big)$$

where we used the induction hypothesis twice, namely for the union of the N $\mathcal{G}$-sets $A_1, \ldots, A_N$ as well as for the N $\mathcal{G}$-sets $A_1 \cap A_{N+1}, \ldots, A_N \cap A_{N+1}$. The induction is complete.

In particular we see that $\mu(F_N) = \nu(F_N)$, $\nu(F_N) \leqslant \nu(G_1) + \ldots + \nu(G_N) < \infty$ by subadditivity, and that (think!) $\mu(G \cap F_N) = \nu(G \cap F_N)$ for any $G \in \mathcal{G}$ (just work out the intersection, similar to the step in the induction....). This shows that on the $\cap$-stable system

$$\tilde{\mathcal{G}} := \{\text{all finite unions of sets in } \quad \mathcal{G}\}$$

μ and ν coincide. Moreover, $\mathcal{G} \subset \tilde{\mathcal{G}} \subset \mathcal{A}$ so that, by assumption $\mathcal{A} = \sigma(\mathcal{G}) \subset \sigma(\tilde{\mathcal{G}}) \subset \sigma(\mathcal{A}) \subset \mathcal{A}$, so that equality prevails in this chain of inclusions. This means that $\tilde{\mathcal{G}}$ is a generator of $\mathcal{A}$ satisfying all the assumptions of Theorem 5.7, and we have reduced everything to this situation.

Problem 5.7 *Intuition:* in two dimensions we have rectangles. Take $I, I' \in \mathcal{J}$. Call the lower left corner of I $a = (a_1, a_2)$, the upper right corner $b = (b_1, b_2)$, and do the same for I' using a', b'. This defines a rectangle uniquely. We are done, if $I \cap I' = \varnothing$. If not (draw a picture!) then we get an overlap which can be described by taking the right-and-upper-most of the two lower left corners a, a' and the left-and-lower-most of the two upper right corners b, b'. That does the trick.

Now rigorously: since $I, I' \in \mathcal{J}$, we have for suitable a_j, b_j, a'_j, b'_j's:

$$I = \mathop{\times}_{j=1}^{n} \big[a_j, b_j\big) \quad \text{and} \quad I' = \mathop{\times}_{j=1}^{n} \big[a'_j, b'_j\big).$$

We want to find $I \cap I'$, or, equivalently the condition under which $x \in I \cap I'$. Now

$$\begin{aligned} x = (x_1, \ldots, x_n) \in I &\iff x_j \in [a_j, b_j) \quad \forall j = 1, 2, \ldots, n \\ &\iff a_j \leqslant x_j < b_j \quad \forall j = 1, 2, \ldots, n \end{aligned}$$

and the same holds for $x \in I'$ (same x, but I'—no typo). Clearly $a_j \leqslant x_j < b_j$, and, at the same time $a'_j \leqslant x_j < b'_j$ holds exactly if

$$\begin{aligned} &\max(a_j, a'_j) \leqslant x_j < \min(b_j, b'_j) \quad \forall j = 1, 2, \ldots, n \\ \iff\ & x \in \mathop{\times}_{j=1}^{n} \big[\max(a_j, a'_j), \min(b_j, b'_j)\big). \end{aligned}$$

This shows that $I \cap I'$ is indeed a 'rectangle', i.e. in $\mathcal{J}$. This could be an empty set (which happens if I and I' do not meet).

Problem 5.8 First we must make sure that $t \cdot B$ is a Borel set if $B \in \mathcal{B}$. We consider first rectangles $I = [[a, b)) \in \mathcal{J}$ where $a, b \in \mathbb{R}^n$. Clearly, $t \cdot I = [[ta, tb))$ where ta, tb are just the scaled vectors. So, scaled rectangles are again rectangles, and therefore Borel sets. Now fix $t > 0$ and set

$$\mathcal{B}_t := \{B \in \mathcal{B}^n \; : \; t \cdot B \in \mathcal{B}^n\}.$$

It is not hard to see that $\mathcal{B}_t$ is itself a σ-algebra and that $\mathcal{J} \subset \mathcal{B}_t \subset \mathcal{B}^n$. But then we get

$$\mathcal{B}^n = \sigma(\mathcal{J}) \subset \sigma(\mathcal{B}_t) = \mathcal{B}_t \subset \mathcal{B}^n,$$

showing that $\mathcal{B}_t = \mathcal{B}^n$, i.e. scaled Borel sets are again Borel sets.

Now define a new measure $\mu(B) := \lambda^n(t \cdot B)$ for Borel sets $B \in \mathcal{B}^n$ (which is, because of the above, well-defined). For rectangles $[\![a, b))$ we get, in particular,

$$\begin{aligned}
\mu[\![a,b)) = \lambda^n\big((t \cdot [\![a,b))\big) &= \lambda^n[\![ta, tb)) \\
&= \prod_{j=1}^n \big((tb_j) - (ta_j)\big) \\
&= \prod_{j=1}^n t \cdot \big(b_j - a_j\big) \\
&= t^n \cdot \prod_{j=1}^n \big(b_j - a_j\big) \\
&= t^n \lambda^n [\![a,b))
\end{aligned}$$

which shows that μ and $t^n\lambda^n$ coincide on the $\cap$-stable generator $\mathcal{J}$ of $\mathcal{B}^n$, hence they're the same everywhere. (Mind the small gap: we should make the mental step that for any measure ν a positive multiple, say, $c \cdot \nu$, is again a measure—this ensures that $t^n\lambda^n$ is a measure, and we need this in order to apply Theorem 5.7. Mind also that we need that μ is finite on all rectangles (obvious!) and that we find rectangles increasing to $\mathbb{R}^n$, e.g. $[-k, k) \times \ldots \times [-k, k)$ as in the proof of Theorem 5.8(ii).)

Problem 5.9 Define $\nu(A) := \mu \circ \theta^{-1}(A)$. Obviously, ν is again a finite measure. Moreover, since $\theta^{-1}(X) = X$, we have

$$\mu(X) = \nu(X) < \infty \quad \text{and, by assumption,} \quad \mu(G) = \nu(G) \quad \forall\, G \in \mathcal{G}.$$

Thus, $\mu = \nu$ on $\mathcal{G}' := \mathcal{G} \cup \{X\}$. Since $\mathcal{G}'$ is a $\cap$-stable generator of $\mathcal{A}$ containing the (trivial) exhausting sequence $X, X, X, \ldots$, the assertion follows from the uniqueness theorem for measures, Theorem 5.7.

Problem 5.10 The necessity of the condition is trivial since $\mathcal{G} \subset \sigma(\mathcal{G}) = \mathcal{B}$, resp., $\mathcal{H} \subset \sigma(\mathcal{H}) = \mathcal{C}$.

Fix $H \in \mathcal{H}$ and define

$$\mu(B) := P(B \cap H) \quad \text{and} \quad \nu(B) := P(B)P(H).$$

Obviously, μ and ν are finite measures on $\mathcal{B}$ having mass $P(H)$ such that μ and ν coincide on the $\cap$-stable generator $\mathcal{G} \cup \{X\}$ of $\mathcal{B}$. Note that this generator contains the exhausting sequence $X, X, X, \ldots$. By the uniqueness theorem for measures, Theorem 5.7, we conclude

$$\mu = \nu \quad \text{on the whole of} \quad \mathcal{B}.$$

Now fix $B \in \mathcal{B}$ and define

$$\rho(C) := P(B \cap C) \quad \text{and} \quad \tau(C) := P(B)P(C).$$

Then the same argument as before shows that $\rho = \sigma$ on $\mathcal{C}$ and, since $B \in \mathcal{B}$ was arbitrary, the claim follows.

6 Existence of measures. Solutions to Problems 6.1–6.11

Problem 6.1 We know already that $\mathcal{B}[0,\infty)$ is a σ-algebra (it is a trace σ-algebra) and, by definition,

$$\Sigma = \{B \cup (-B) \;:\; B \in \mathcal{B}[0,\infty)\}$$

if we write $-B := \{-b \;:\; b \in \mathcal{B}[0,\infty)\}$.

Since the structure $B \cup (-B)$ is stable under complementation and countable unions it is clear that Σ is indeed a σ-algebra.

One possibility to extend μ defined on Σ would be to take $B \in \mathcal{B}(\mathbb{R})$ and define $B^+ := B \cap [0,\infty)$ and $B^- := B \cap (-\infty, 0)$ and to set

$$\nu(B) := \mu(B^+ \cup (-B^+)) + \mu((-B^-) \cup B^-)$$

which is obviously a measure. We cannot expect uniqueness of this extension since Σ does not generate $\mathcal{B}(\mathbb{R})$—not all Borel sets are symmetric.

Problem 6.2 By definition we have

$$\mu^*(Q) = \inf\left\{\sum_j \mu(B_j) \;:\; (B_j)_{j\in\mathbb{N}} \subset \mathcal{A},\; \bigcup_{j\in\mathbb{N}} B_j \supset Q\right\}.$$

(i) Assume first that $\mu^*(Q) < \infty$. By the definition of the infimum we find for every $\epsilon > 0$ a sequence $(B_j^\epsilon)_{j\in\mathbb{N}} \subset \mathcal{A}$ such that $B^\epsilon := \bigcup_j B_j^\epsilon \supset Q$ and, because of σ-subadditivity,

$$\mu(B^\epsilon) - \mu^*(Q) \leqslant \sum_j \mu(B_j^\epsilon) - \mu^*(Q) \leqslant \epsilon.$$

Set $B := \bigcap_k B^{1/k} \in \mathcal{A}$. Then $B \supset Q$ and $\mu(B) = \mu^*(B) = \mu^*(Q)$.

Now let $N \in \mathcal{A}$ and $N \subset B \setminus Q$. Then

$$\begin{aligned} B \setminus N \supset B \setminus (B \setminus Q) = B \cap [(B \cap Q^c)^c] &= B \cap [B^c \cup Q] \\ &= B \cap Q \\ &= Q. \end{aligned}$$

So,

$$\mu^*(Q) - \mu(N) = \mu(B) - \mu(N) = \mu(B \setminus N) = \mu^*(B \setminus N) \geqslant \mu^*(Q)$$

which means that $\mu(N) = 0$.

If $\mu^*(Q) = \infty$, we take the exhausting sequence $(A_j)_{j\in\mathbb{N}} \subset \mathcal{A}$ with $A_j \uparrow X$ and $\mu(A_j) < \infty$ and set $Q_j := A_j \cap Q$ for every $j \in \mathbb{N}$. By the first part we can find sets $C_j \in \mathcal{A}$ with $C_j \supset Q_j$, $\mu(C_j) = \mu^*(Q_j)$ and $\mu(N) = 0$ for all $N \in \mathcal{A}$ with $N \subset C_j \setminus Q_j$. Without loss of generality we can assume that $C_j \subset A_j$, otherwise we replace C_j by $A_j \cap C_j$. Indeed, $C_j \cap A_j \supset Q_j$, $C_j \cap A_j \in \mathcal{A}$,

$$\mu^*(Q_j) = \mu(C_j) \geqslant \mu(A_j \cap C_j) \geqslant \mu^*(Q_j)$$

and $C_j \setminus Q_j \supset (C_j \cap A_j) \setminus Q_j$, i.e. we have again that all measurable $N \subset (C_j \cap A_j) \setminus Q_j$ satisfy $\mu(N) = 0$.

Assume now that $N \subset C \setminus Q$ and $N \in \mathcal{A}$. Then $N_j := N \cap A_j \in \mathcal{A}$ and

$$N_j = N \cap A_j \subset (C \setminus Q) \cap A_j = C_j \setminus Q = C_j \setminus Q_j.$$

Thus $\mu(N_j) = 0$ and, by σ-subadditivity, $\mu(N) \leqslant \sum_{j=1}^\infty \mu(N_j) = 0$.

(ii) Define $\bar\mu := \mu^*\big|_{\mathcal{A}^*}$. We know from Theorem 6.1 that $\bar\mu$ is a measure on $\mathcal{A}^*$ and, because of the monotonicity of μ^*, we know that for all $N^* \in \mathcal{A}^*$ with $\bar\mu(N^*)$ we have

$$\forall\, M \subset N^* \,:\, \mu^*(M) \leqslant \mu^*(N^*) = \bar\mu(N^*) = 0.$$

It remains to show that $M \in \mathcal{A}^*$. Because of (6.4) we have to show that

$$\forall\, Q \subset X \,:\, \mu^*(Q) = \mu^*(Q \cap M) + \mu(Q \setminus M).$$

Since μ^* is subadditive we find for all $Q \subset X$

$$\begin{aligned}\mu^*(Q) &= \mu^*\big((Q \cap M) \cup (Q \setminus M)\big)\\ &\leqslant \mu^*(Q \cap M) + \mu^*(Q \setminus M)\\ &= \mu^*(Q \setminus M)\\ &\leqslant \mu^*(Q),\end{aligned}$$

which means that $M \in \mathcal{A}^*$.

(iii) Obviously, $(X, \mathcal{A}^*, \bar\mu)$ extends $(X, \mathcal{A}, \mu)$ since $\mathcal{A} \subset \mathcal{A}^*$ and $\bar\mu\big|_{\mathcal{A}} = \mu$. In view of Problem 4.13 we have to show that

$$\mathcal{A}^* = \{A \cup N \,:\, A \in \mathcal{A},\quad N \in \mathfrak{N}\} \tag{$*$}$$

with $\mathfrak{N} = \{N \subset X \,:\, N$ is subset of an $\mathcal{A}$-measurable null set or, alternatively,

$$\mathcal{A}^* = \{A^* \subset X \,:\, \exists\, A, B \in \mathcal{A},\ A \subset A^* \subset B,\ \mu(B \setminus A) = 0\}. \tag{$**$}$$

We are going to use both equalities and show '$\supset$' in $(*)$ and '$\subset$' in $(**)$ (which is enough since, cf. Problem 4.13 asserts the equality of the right-hand sides of $(*)$, $(**)$!).

'$\supset$': By part (ii), subsets of $\mathcal{A}$-null sets are in $\mathcal{A}^*$ so that every set of the form $A \cup N$ with $A \in \mathcal{A}$ and N being a subset of an $\mathcal{A}$ null set is in $\mathcal{A}^*$.

'$\subset$': By part (i) we find for every $A^* \in \mathcal{A}^*$ some $A \in \mathcal{A}$ such that $A \supset A^*$ and $A \setminus A^*$ is an $\mathcal{A}^*$ null set. By the same argument we get $B \in \mathcal{A}$, $B \supset (A^*)^c$ and $B \setminus (A^*)^c = B \cap A^* = A^* \setminus B^c$ is an $\mathcal{A}^*$ null set. Thus,

$$B^c \subset A^* \subset A$$

and

$$A \setminus B^c \subset \big(A \setminus A^*\big) \cup \big(A^* \setminus B^c\big) = \big(A \setminus A^*\big) \cup \big(B \setminus (A^*)^c\big)$$

which is the union of two $\mathcal{A}^*$ null sets, i.e. $A \setminus B^c$ is an $\mathcal{A}$ null set.

Problem 6.3 (i) A little geometry first: a solid, open disk of radius r, centre 0 is the set $B_r(0) := \{(x,y) \in \mathbb{R}^2 \ : \ x^2 + y^2 < r^2\}$. Now the n-dimensional analogue is clearly $\{x \in \mathbb{R}^n \ : \ x_1^2 + x_2^2 + \ldots + x_n^2 < r^2\}$ (including $n = 1$ where it reduces to an interval). We want to inscribe a box into a ball.

Claim: $Q_\epsilon(0) := \bigtimes_{j=1}^{n} \big[-\frac{\epsilon}{\sqrt{n}}, \frac{\epsilon}{\sqrt{n}}\big) \subset B_{2\epsilon}(0)$. Indeed,

$$\begin{aligned} x \in Q_\epsilon(0) &\implies x_1^2 + x_2^2 + \ldots + x_n^2 \leqslant \frac{\epsilon^2}{n} + \frac{\epsilon^2}{n} + \ldots + \frac{\epsilon^2}{n} < (2\epsilon)^2 \\ &\implies x \in B_{2\epsilon}(0), \end{aligned}$$

and the claim follows.

Observe that $\lambda^n(Q_\epsilon(0)) = \prod_{j=1}^{n} \frac{2\epsilon}{\sqrt{n}} > 0$. Now take some open set U. By translating it we can achieve that $0 \in U$ and, as we know, this movement does not affect $\lambda^n(U)$. As $0 \in U$ we find some $\epsilon > 0$ such that $B_\epsilon(0) \subset U$, hence

$$\lambda^n(U) \geqslant \lambda^n(B_\epsilon(0)) \geqslant \lambda(Q_\epsilon(0)) > 0.$$

(ii) For closed sets this is, *in general*, wrong. Trivial counterexample: the singleton $\{0\}$ is closed, it is Borel (take a countable sequence of nested rectangles, centered at 0 and going down to $\{0\}$) and the Lebesgue measure is zero.

To get strictly positive Lebesgue measure, one possibility is to have interior points, i.e. closed sets which have non-empty interior do have positive Lebesgue measure.

Problem 6.4 (i) Without loss of generality we can assume that $a < b$. We have $[a+\frac{1}{k}, b) \uparrow (a,b)$ as $k \to \infty$. Thus, by the continuity of measures, Theorem 4.4, we find (write $\lambda = \lambda^1$, for short)

$$\lambda(a,b) = \lim_{k\to\infty} \lambda\left[a + \frac{1}{k}, b\right) = \lim_{k\to\infty} \left(b - a - \frac{1}{k}\right) = b - a.$$

Since $\lambda[a,b) = b - a$, too, this proves again that

$$\lambda(\{a\}) = \lambda([a,b) \setminus (a,b)) = \lambda[a,b) - \lambda(a,b) = 0.$$

(ii) The hint says it all: H is contained in the union $y + \bigcup_{k\in\mathbb{N}} A_k$ for some y and we have $\lambda^2(A_k) = (2\epsilon\, 2^{-k}) \cdot (2k) = 4 \cdot \epsilon \cdot k2^{-k}$. Using the σ-subadditivity and monotonicity of measures (the A_k's are clearly not disjoint) as well as the translational invariance of the Lebesgue measure we get

$$0 \leqslant \lambda^2(H) \leqslant \lambda^2\left(\bigcup_{k=1}^{\infty} A_k\right) \leqslant \sum_{k=1}^{\infty} \lambda(A_k) = \sum_{k=1}^{\infty} 4 \cdot \epsilon \cdot k2^{-k} = C\epsilon$$

where C is the finite (!) constant $4\sum_{k=1}^{\infty} k2^{-k}$ (check convergence!). As ϵ was arbitrary, we can let it $\to 0$ and the claim follows.

(iii) *n-dimensional version of (i):* We have $I = \mathop{\times}_{j=1}^{n} (a_j, b_j)$. Set $I_k := \mathop{\times}_{j=1}^{n} [a_j + \frac{1}{k}, b_j)$. Then $I_k \uparrow I$ as $k \to \infty$ and we have (write $\lambda = \lambda^n$, for short)

$$\lambda(I) = \lim_{k\to\infty} \lambda(I_k) = \lim_{k\to\infty} \prod_{j=1}^{n} \left(b_j - a_j - \frac{1}{k}\right) = \prod_{j=1}^{n} (b_j - a_j)\,.$$

n-dimensional version of (ii): The changes are obvious: $A_k = [-\epsilon 2^{-k}, \epsilon 2^{-k}) \times [-k, k)^{n-1}$ and $\lambda^n(A_k) = 2^n \cdot \epsilon \cdot 2^{-k} \cdot k^{n-1}$. The rest stays as before, since the sum $\sum_{k=1}^{\infty} k^{n-1} 2^{-k}$ still converges to a finite value.

Problem 6.5 (i) All we have to show is that $\lambda^1(\{x\}) = 0$ for any $x \in \mathbb{R}$. But this has been shown already in problem 6.3(i).

(ii) Take the Dirac measure: δ_0. Then $\{0\}$ is an atom as $\delta_0(\{0\}) = 1$.

(iii) Let C be countable and let $\{c_1, c_2, c_3, \ldots\}$ be an enumeration (could be finite, if C is finite). Since singletons are in $\mathcal{A}$, so is C as a countable union of the sets $\{c_j\}$. Using the σ-additivity of a measure we get

$$\mu(C) = \mu(\cup_{j\in\mathbb{N}}\{c_j\}) = \sum_{j\in\mathbb{N}} \mu(\{c_j\}) = \sum_{j\in\mathbb{N}} 0 = 0.$$

(iv) If $y_1, y_2, \ldots, y_N$ are atoms of mass $P(\{y_j\}) \geqslant \frac{1}{k}$ we find by the additivity and monotonicity of measures

$$\begin{aligned} \frac{N}{k} &\leqslant \sum_{j=1}^{N} P(\{x_j\}) \\ &= P\left(\bigcup_{j=1}^{N} \{y_j\}\right) \\ &= P(\{y_1, \ldots, y_N\}) \leqslant P(\mathbb{R}) = 1 \end{aligned}$$

so $\frac{N}{k} \leqslant 1$, i.e. $N \leqslant k$, and the claim in the hint (about the maximal number of atoms of given size) is shown.

Now denote, as in the hint, the atoms with measure of size $[\frac{1}{k}, \frac{1}{k-1})$ by $y_1^{(k)}, \ldots y_{N(k)}^{(k)}$ where $N(k) \leqslant k$ is their number. Since

$$\bigcup_{k\in\mathbb{N}} [\tfrac{1}{k}, \tfrac{1}{k-1}) = (0, \infty)$$

we exhaust all possible sizes for atoms.

There are at most countably many (actually: finitely many) atoms in each size range. Since the number of size ranges is countable and since countably many countable sets make up a countable set, we can relabel the atoms as $x_1, x_2, x_3, \dots$ (could be finite) and, as we have seen in exercise 4.6(ii), the set-function

$$\nu := \sum_j P(\{x_j\}) \cdot \delta_{x_j}$$

(no matter whether the sum is over a finite or countably infinite set of j's) is indeed a measure on $\mathbb{R}$. But more is true: for any Borel set A

$$\begin{aligned}\nu(A) &= \sum_j P(\{x_j\}) \cdot \delta_{x_j}(A) \\ &= \sum_{j\,:\,x_j \in A} P(\{x_j\}) \\ &= P(A \cap \{x_1, x_2, \dots\}) \leqslant P(A)\end{aligned}$$

showing that $\mu(A) := P(A) - \nu(A)$ is a positive number for each Borel set $A \in \mathcal{B}$. This means that $\mu : \mathcal{B} \to [0, \infty]$. Let us check M_1 and M_2. Using M_1, M_2 for P and ν (for them they are clear, as P, ν are measures!) we get

$$\mu(\emptyset) = P(\emptyset) - \nu(\emptyset) = 0 - 0 = 0$$

and for a disjoint sequence $(A_j)_{j \in \mathbb{N}} \subset \mathcal{B}$ we have

$$\begin{aligned}\mu\Big(\bigcup_j A_j\Big) &= P\Big(\bigcup_j A_j\Big) - \nu\Big(\bigcup_j A_j\Big) \\ &= \sum_j P(A_j) - \sum_j \nu(A_j) \\ &= \sum_j \big(P(A_j) - \nu(A_j)\big) \\ &= \sum_j \mu(A_j)\end{aligned}$$

which is M_2 for μ.

Problem 6.6 (i) Fix a sequence of numbers $\epsilon_k > 0, k \in \mathbb{N}_0$ such that $\sum_{k \in \mathbb{N}_0} \epsilon_k < \infty$. For example we could take a geometric series with general term $\epsilon_k := 2^{-k}$. Now define *open* intervals $I_k := (k - \epsilon_k, k + \epsilon_k)$, $k \in \mathbb{N}_0$ (these are open sets!) and call their union $I := \bigcup_{k \in \mathbb{N}_0} I_k$. As countable union of open sets I is again open. Using the σ-(sub-)additivity of $\lambda = \lambda^1$ we find

$$\lambda(I) = \lambda\Big(\bigcup_{k \in \mathbb{N}_0} I_k\Big) \overset{(*)}{\leqslant} \sum_{k \in \mathbb{N}_0} \lambda(I_k) = \sum_{k \in \mathbb{N}_0} 2\epsilon_k = 2 \sum_{k \in \mathbb{N}_0} \epsilon_k < \infty.$$

By 6.4(i), $\lambda(I) > 0$.

Note that in step $(*)$ equality holds (i.e. we would use σ-additivity rather than σ-subadditivity) if the I_k are pairwise disjoint. This happens, if all $\epsilon_k < \frac{1}{2}$ (think!), but to be on the safe side and in order not to have to worry about such details we use sub-additivity.

(ii) Take the open interior of the sets A_k, $k \in \mathbb{N}$, from the hint to 6.4(ii). That is, take the open rectangles $B_k := (-2^{-k}, 2^{-k}) \times (-k, k)$, $k \in \mathbb{N}$, (we choose $\epsilon = 1$ since we are after *finiteness* and not necessarily *smallness*). That these are open sets will be seen below. Now set $B = \bigcup_{k\in\mathbb{N}} B_k$ and observe that the union of open sets is always open. B is also unbounded and it is geometrically clear that B is connected as it is some kind of lozenge-shaped 'staircase' (draw a picture!) around the y-axis. Finally, by σ-subadditivity and using 6.4(ii) we get

$$\begin{aligned}\lambda^2(B) = \lambda^2\Bigg(\bigcup_{k\in\mathbb{N}} B_k\Bigg) &\leqslant \sum_{k\in\mathbb{N}} \lambda^2(B_k)\\ &= \sum_{k\in\mathbb{N}} 2\cdot 2^{-k}\cdot 2\cdot k\\ &= 4\sum_{k\in\mathbb{N}} k\cdot 2^{-k} < \infty.\end{aligned}$$

It remains to check that an open rectangle is an open set. For this take any open rectangle $R = (a,b)\times(c,d)$ and pick $(x,y)\in R$. Then we know that $a < x < b$ and $c < y < d$ and since we have strict inequalities, we have that the smallest distance of this point to any of the four boundaries (draw a picture!) $h := \min\{|a-x|, |b-x|, |c-y|, |d-y|\} > 0$. This means that a square around (x,y) with side-length $2h$ is inside R and what we're going to do is to inscribe into this virtual square an open disk with radius h and centre (x,y). Since the circle is again in R, we are done. The equation for this disk is

$$(x',y') \in B_h(x,y) \iff (x-x')^2 + (y-y')^2 < h^2$$

Thus,

$$\begin{aligned}|x'-x| &\leqslant \sqrt{|x-x'|^2 + |y-y'|^2} < h\\ \text{and}\quad |y'-y| &\leqslant \sqrt{|x-x'|^2 + |y-y'|^2} < h\end{aligned}$$

i.e. $x-h < x' < x+h$ and $y-h < y' < y+h$ or $(x',y') \in (x-h,x+h)\times(y-h,y+h)$, which means that (x',y') is in the rectangle of sidelength $2h$ centered at (x,y). since (x',y') was an arbitrary point of $B_h(x,y)$, we are done.

(iii) No, this is impossible. Since we are in one dimension, connectedness forces us to go between points in a straight, uninterrupted line. Since the set is unbounded, this means that we must have a line of the sort (a,∞) or $(-\infty,b)$ in our set and in both cases Lebesgue measure is infinite. In all dimensions $n > 1$, see part (ii) for two dimensions, we can, however, construct connected, unbounded open sets with finite Lebesgue measure.

Problem 6.7 Fix $\epsilon > 0$ and let $\{q_j\}_{j\in\mathbb{N}}$ be an enumeration of $\mathbb{Q}\cap[0,1]$. Then

$$U := U_\epsilon := \bigcup_{j\in\mathbb{N}} \left(q_j - \epsilon 2^{-j-1}, q_j + \epsilon 2^{-j-1}\right) \cap [0,1]$$

is a dense open set in $[0,1]$ and, because of σ-subadditivity,

$$\lambda(U) \leqslant \sum_{j\in\mathbb{N}} \lambda\left(q_j - \epsilon 2^{-j-1}, q_j + \epsilon 2^{-j-1}\right) = \sum_{j\in\mathbb{N}} \frac{\epsilon}{2^j} = \epsilon.$$

Problem 6.8 Assume first that for every $\epsilon > 0$ there is some open set $U_\epsilon \supset N$ such that $\lambda(U_\epsilon) \leqslant \epsilon$. Then

$$\lambda(N) \leqslant \lambda(U_\epsilon) \leqslant \epsilon \quad \forall \epsilon > 0,$$

which means that $\lambda(N) = 0$.

Conversely, let $\lambda^*(N) = \inf\left\{\sum_j \lambda(U_j) \;:\; U_j \in \mathcal{O},\ \cup_{j\in\mathbb{N}} U_j \supset N\right\}$. Since for the Borel set N we have $\lambda^*(N) = \lambda(N) = 0$, the definition of the infimum guarantees that for every $\epsilon > 0$ there is a sequence of open sets $(U_j^\epsilon)_{j\in\mathbb{N}}$ covering N, i.e. such that $U^\epsilon := \bigcup_j U_j^\epsilon \supset N$. Since U^ϵ is again open we find because of σ-subadditivity

$$\lambda(N) \leqslant \lambda(U^\epsilon) = \lambda\left(\bigcup_j U_j^\epsilon\right) \leqslant \sum_j \lambda(U_j^\epsilon) \leqslant \epsilon.$$

Attention: A construction along the lines of Problem 3.12, hint to part (ii), using open sets $U^\delta := N + B_\delta(0)$ is, in general not successful:

- it is not clear that U^δ has finite Lebesgue measure (o.k. one can overcome this by considering $N \cap [-k,k]$ and then letting $k \to \infty$...)
- $U^\delta \downarrow \bar{N}$ and *not* N (unless N is closed, of course). If, say, N is a dense set of $[0,1]$, this approach leads nowhere.

Problem 6.9 Observe that the sets $C_k := \bigcup_{j=k}^\infty A_j$, $k \in \mathbb{N}$, decrease as $k \to \infty$—we admit less and less sets in the union, i.e. the union becomes smaller. Since P is a probability measure, $P(C_k) \leqslant 1$ and therefore Theorem 4.4(iii') applies and shows that

$$P\left(\bigcap_{k=1}^\infty \bigcup_{j=k}^\infty A_j\right) = P\left(\bigcap_{k=1}^\infty C_k\right) = \lim_{k\to\infty} P(C_k).$$

On the other hand, we can use σ-subadditivity of the measure P to get

$$P(C_k) = P\left(\bigcup_{j=k}^\infty A_j\right) \leqslant \textstyle\sum_{j=k}^\infty P(A_j)$$

but this is the tail of the convergent (!) sum $\sum_{j=1}^\infty P(A_j)$ and, as such, it goes to zero as $k \to \infty$. Putting these bits together, we see

$$P\left(\bigcap_{k=1}^\infty \bigcup_{j=k}^\infty A_j\right) = \lim_{k\to\infty} P(C_k) \leqslant \lim_{k\to\infty} \sum_{j=k}^\infty P(A_j) = 0,$$

and the claim follows.

Problem 6.10 (i) We can work out the 'optimal' $\mathcal{A}$-cover of (a,b):

Case 1: $a,b\in[0,1)$. Then $[0,1)$ is the best possible cover of (a,b), thus $\mu^*(a,b)=\mu[0,1)=\frac{1}{2}$.

Case 2: $a,b\in[1,2)$. Then $[1,2)$ is the best possible cover of (a,b), thus $\mu^*(a,b)=\mu[1,2)=\frac{1}{2}$.

Case 3: $a\in[0,1), b\in[1,2)$. Then $[0,1)\cup[1,2)$ is the best possible cover of (a,b), thus $\mu^*(a,b)=\mu[0,1)+\mu[1,2)=1$.

And in the case of a singleton $\{a\}$ the best possible cover is always either $[0,1)$ or $[1,2)$ so that $\mu^*(\{a\})=\frac{1}{2}$ for all a.

(ii) Assume that $(0,1)\in\mathcal{A}^*$. Since $\mathcal{A}\subset\mathcal{A}^*$, we have $[0,1)\in\mathcal{A}^*$, hence $\{0\}=[0,1)\smallsetminus(0,1)\in\mathcal{A}^*$. Since $\mu^*(0,1)=\mu^*(\{0\})=\frac{1}{2}$, and since μ^* is a measure on $\mathcal{A}^*$ (cf. step 4 in the proof of Theorem 6.1), we get

$$\frac{1}{2}=\mu[0,1)=\mu^*[0,1)=\mu^*(0,1)+\mu^*\{0\}=\frac{1}{2}+\frac{1}{2}=1$$

leading to a contradiction. Thus neither $(0,1)$ nor $\{0\}$ are elements of $\mathcal{A}^*$.

Problem 6.11 Since $\mathcal{A}\subset\mathcal{A}^*$, the only interesting sets (to which one could extend μ) are those $B\subset\mathbb{R}$ where both B and B^c are uncountable. By definition,

$$\gamma^*(B)=\inf\Big\{\sum_j\gamma(A_j)\;:\;A_j\in\mathcal{A},\;\bigcup_j A_j\supset B\Big\}.$$

The infimum is obviously attained for $A_j=\mathbb{R}$, so that $\gamma^*(B)=\gamma^*(B^c)=1$. On the other hand, since γ^* is necessarily additive on $\mathcal{A}^*$, the assumption that $B\in\mathcal{A}^*$ leads to a contradiction:

$$1=\gamma(\mathbb{R})=\gamma^*(\mathbb{R})=\gamma^*(B)+\gamma^*(B^c)=2.$$

Thus, $\mathcal{A}=\mathcal{A}^*$.

7 Measurable mappings. Solutions to Problems 7.1–7.11

Problem 7.1 We have $\tau_x^{-1}(z) = z + x$. According to Lemma 7.2 we have to check that

$$\tau_x^{-1}([a,b)) \in \mathcal{B}^n \qquad \forall\, [a,b) \in \mathcal{J}$$

since the rectangles $\mathcal{J}$ generate $\mathcal{B}^n$. Clearly,

$$\tau_x^{-1}([a,b)) = [a,b) + x = [a+x, b+x) \in \mathcal{J} \subset \mathcal{B}^n,$$

and the claim follows.

Problem 7.2 We had $\Sigma' = \{A' \subset X' : T^{-1}(A') \in \mathcal{A}\}$ where $\mathcal{A}$ was a σ-algebra of subsets of X. Let us check the properties (Σ_1)–(Σ_3).

(Σ_1) Take $\varnothing \subset X'$. Then $T^{-1}(\varnothing) = \varnothing \in \mathcal{A}$, hence $\varnothing \in \Sigma'$.

(Σ_2) Take any $B \in \Sigma'$. Then $T^{-1}(B) \in \mathcal{A}$ and therefore $T^{-1}(B^c) = \big(T^{-1}(B)\big)^c \in \mathcal{A}$ since all set-operations interchange with inverse maps and since $\mathcal{A}$ is a σ-algebra. This shows that $B^c \in \Sigma'$.

(Σ_3) Take any sequence $(B_j)_{j\in\mathbb{N}} \subset \Sigma'$. Then, using again the fact that $\mathcal{A}$ is a σ-algebra, $T^{-1}(\cup_j B_j) = \bigcup_j T^{-1}(B_j) \in \mathcal{A}$ which proves that $\bigcup_j B_j \in \Sigma'$.

Problem 7.3 (i) First of all we remark that $T_i^{-1}(\mathcal{A}_i)$ is itself a σ-algebra, cf. Example 3.3(vii).

If $\mathcal{C}$ is a σ-algebra of subsets of X such that $T_i : (X, \mathcal{C}) \to (X_i, \mathcal{A}_i)$ becomes measurable, we know from the very definition that $T^{-1}(\mathcal{A}_i) \subset \mathcal{C}$. From this, however, it is clear that $T^{-1}(\mathcal{A}_i)$ is the minimal σ-algebra that renders T measurable.

(ii) From part (i) we know that $\sigma(T_i, i \in I)$ necessarily contains $T_i^{-1}(\mathcal{A}_i)$ for every $i \in I$. Since $\bigcup_i T_i^{-1}(\mathcal{A}_i)$ is, in general, not a σ-algebra, we have $\sigma\Big(\bigcup_i T_i^{-1}(\mathcal{A}_i)\Big) \subset \sigma(T_i, i \in I)$. On the other hand, each T_i is, because of $T_i^{-1}(\mathcal{A}_i) \subset \bigcup_i T_i^{-1}(\mathcal{A}_i) \subset \sigma(T_i, i \in I)$ measurable w.r.t. $\sigma\Big(\bigcup_i T_i^{-1}(\mathcal{A}_i)\Big)$ and this proves the claim.

Problem 7.4 We have to show that

$$\begin{aligned} & f : (F, \mathcal{F}) \to (X, \sigma(T_i,\ i \in I)) \quad \text{measurable} \\ \iff \quad & \forall\, i \in I\ :\ T_i \circ f : (F, \mathcal{F}) \to (X_i, \mathcal{A}_i) \quad \text{measurable.} \end{aligned}$$

Now

$$\begin{aligned}
\forall\, i \in I \,:\, (T_i \circ f)^{-1}(\mathcal{A}_i) \subset \mathcal{F} &\iff \forall\, i \in I \,:\, f^{-1}\big(T_i^{-1}(\mathcal{A}_i)\big) \subset \mathcal{F}\\
&\iff f^{-1}\Big(\bigcup_{i\in I} T_i^{-1}(\mathcal{A}_i)\Big) \subset \mathcal{F}\\
&\overset{(*)}{\iff} \sigma\Big[f^{-1}\Big(\bigcup_{i\in I} T_i^{-1}(\mathcal{A}_i)\Big)\Big] \subset \mathcal{F}\\
&\overset{(**)}{\iff} f^{-1}\Big(\sigma\Big[\bigcup_{i\in I} T_i^{-1}(\mathcal{A}_i)\Big]\Big) \subset \mathcal{F}.
\end{aligned}$$

Only (*) and (**) are not immediately clear. The direction '$\Longleftarrow$' in (*) is trivial, while '$\Longrightarrow$' follows if we observe that the right-hand side, $\mathcal{F}$, is a σ-algebra. The equivalence (**) is another case of Problem 7.8 (see there for the solution!).

Problem 7.5 Using the notation of the foregoing Problem 7.4 we put $I = \{1, 2, \dots, m\}$, $T_j := \pi_j : \mathbb{R}^m \to \mathbb{R}$, $\pi_j(x_1, \dots, x_m) := x_j$ is the coordinate projection, $\mathcal{A}_j := \mathcal{B}(\mathbb{R})$. Since each π_j is continuous, we have $\sigma(\pi_1, \dots, \pi_m) \subset \mathcal{B}(\mathbb{R}^m)$ so that Problem 7.4 applies and proves

$$\begin{gathered}
f \text{ is } \mathcal{B}(\mathbb{R}^m)\text{-measurable} \iff\\
f_j = \pi_j \circ f \text{ is } \mathcal{B}(\mathbb{R})\text{-measurable for all } j = 1, 2, \dots, m.
\end{gathered}$$

Remark. We will see, in fact, in Chapter 13 (in particular in Theorem 13.10) that we have the equality $\sigma(\pi_1, \dots, \pi_m) = \mathcal{B}(\mathbb{R}^m)$.

Problem 7.6 In general the direct image $T(\mathcal{A})$ of a σ-algebra is not any longer a σ-algebra. (Σ_1) and (Σ_3) hold, but (Σ_2) will, in general, fail. Here is an example: Take $X = X' = \mathbb{N}$, take any σ-algebra $\mathcal{A}$ other than $\{\emptyset, \mathbb{N}\}$ in $\mathbb{N}$, and let $T : \mathbb{N} \to \mathbb{N}$, $T(j) = 1$ be the constant map. Then $T(\emptyset) = \emptyset$ but $T(A) = \{1\}$ whenever $A \neq \emptyset$. Thus, $\{1\} = T(A^c) \neq [T(A)]^c = \mathbb{N} \setminus \{1\}$ but equality would be needed if $T(\mathcal{A})$ were a σ-algebra. This means that Σ_2 fails.

Necessary and sufficient for $T(\mathcal{A})$ to be a σ-algebra is, clearly, that T^{-1} is a measurable map $T^{-1} : X' \to X$.

Problem 7.7 Consider for $t > 0$ the dilation $m_t : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto t \cdot x$. Since m_t is continuous, it is Borel measurable. Moreover, $m_t^{-1} = m_{1/t}$ and so

$$t \cdot B = m_{1/t}^{-1}(B)$$

which shows that $\lambda^n(t \cdot B) = \lambda^n \circ m_{1/t}^{-1}(B) = m_{1/t}(\lambda^n)(B)$ is actually an image measure of λ^n. Now show the formula first for rectangles $B = \bigtimes_{j=1}^{n} [a_j, b_j)$ (as in Problem 5.8) and deduce the statement from the uniqueness theorem for measures.

Problem 7.8 We have

$$T^{-1}(\mathcal{G}) \subset \underbrace{T^{-1}(\sigma(\mathcal{G}))}_{\text{is itself a } \sigma\text{-algebra}} \implies \sigma(T^{-1}(\mathcal{G})) \subset T^{-1}(\sigma(\mathcal{G})).$$

For the converse consider $T : (X, \sigma(T^{-1}(\mathcal{G}))) \to (Y, \sigma(\mathcal{G}))$. By the very choice of the σ-algebras and since $T^{-1}(\mathcal{G}) \subset \sigma(T^{-1}(\mathcal{G}))$ we find that T is $\sigma(T^{-1}(\mathcal{G}))/\sigma(\mathcal{G})$ measurable—mind that we only have to check measurability at a generator (here: $\mathcal{G}$) in the image region. Thus,

$$T^{-1}(\sigma(\mathcal{G})) \subset \sigma(T^{-1}(\mathcal{G})).$$

Problem 7.9 (i) **Note the misprint:** we need to assume that $\mu[-n, n) < \infty$ for all $n \in \mathbb{N}$.

Monotonicity: If $x \leqslant 0 \leqslant y$, then $F_\mu(x) \leqslant 0 \leqslant F_\mu(y)$.

If $0 < x \leqslant y$, we have $[0, x) \subset [0, y)$ and so $0 \leqslant F_\mu(x) = \mu[0, x) \leqslant \mu[0, y) = F_\mu(y)$.

If $x \leqslant y < 0$, we have $[y, 0) \subset [x, 0)$ and so $0 \leqslant -F_\mu(y) = \mu[y, 0) \leqslant \mu[x, 0) = -F_\mu(x)$, i.e. $F_\mu(x) \leqslant F_\mu(y) \leqslant 0$.

Left-continuity: Let us deal with the case $x \geqslant 0$ only, the case $x < 0$ is analogous (and even easier). Assume first that $x > 0$. Take any sequence $x_k < x$ and $x_k \uparrow x$ as $k \to \infty$. Without loss of generality we can assume that $0 < x_k < x$. Then $[0, x_k) \uparrow [0, x)$ and using Theorem 4.4(iii') implies

$$\lim_{k\to\infty} F_\mu(x_k) = \lim_{k\to\infty} \mu[0, x_k) = \mu[0, x) = F_\mu(x).$$

If $x = 0$ we must take a sequence $x_k < 0$ and we have then $[x_k, 0) \downarrow [0, 0) = \varnothing$. Again by Theorem 4.4, now (iii''), we get

$$\lim_{k\to\infty} F_\mu(x_k) = -\lim_{k\to\infty} \mu[x_k, 0) = \mu(\varnothing) = 0 = F_\mu(0).$$

which shows left-continuity at this point, too.

We remark that, since for a sequence $y_k \downarrow y$, $y_k > y$ we have $[0, y_k) \downarrow [0, y]$, and not $[0, y)$, we cannot expect right-continuity in general.

(ii) Since $\mathcal{J} = \{[a, b), a \leqslant b\}$ is a semi-ring (cf. Proposition 6.4) it is enough to check that ν_F is a premeasure on $\mathcal{J}$. This again amounts to showing (M_1) and (M_2) relative to $\mathcal{J}$ (mind you: ν_F is not a *measure* as $\mathcal{J}$ is not a σ-algebra....). We do this in the equivalent form of Theorem 4.4, i.e. we prove (i), (ii) and (iii') of Theorem 4.4:

(i) $\nu_F(\varnothing) = \nu_F[a, a) = F(a) - F(a) = 0$ for any a.

(ii) Let $a \leqslant b \leqslant c$ so that $[a, b), [b, c) \in \mathcal{J}$ are disjoint sets and $[a, c) = [a, b) \cup [b, c) \in \mathcal{J}$ (the latter is crucial). Then we have

$$\begin{aligned}
\nu_F[a, b) + \nu_F[b, c) &= F(b) - F(a) + F(c) - F(b) \\
&= F(c) - F(a) \\
&= \nu_F[a, c) \\
&= \nu_F\big([a, b) \cup [b, c)\big).
\end{aligned}$$

(iii') (Sufficient since ν_F is finite for every set $[a,b)$). Now take a sequence of intervals $[a_k, b_k)$ which decreases towards some $[a,b) \in \mathcal{J}$. This means that $a_k \uparrow a$, $a_k \leqslant a$ and $b_k \downarrow b$, $b_k \geqslant b$ because the intervals are nested (gives increasing-decreasing sequences). If $b_k > b$ for infinitely many k, this would mean that $[a_k, b_k) \to [a,b] \notin \mathcal{J}$ since $b \in [a_k, b_k)$ for all k. Since we are only interested in sequences whose limits stay in $\mathcal{J}$, the sequence b_k must reach b after finitely many steps and stay there to give $[a,b)$. Thus, we may assume directly that we have only $[a_k, b) \downarrow [a,b)$ with $a_k \uparrow a$, $a_k \leqslant a$. But then we can use left-continuity and get

$$\begin{aligned}\lim_{k\to\infty} \nu_F[a_k, b) = \lim_{k\to\infty} (F(b) - F(a_k)) &= F(b) - F(a)\\ &= \nu_F[a,b).\end{aligned}$$

Note that ν_F takes on only positive values because F increases.

This means that we find *at least one* extension. Uniqueness follows since $\nu_F[-k,k) = F(k) - F(-k) < \infty$ and $[-k,k) \uparrow \mathbb{R}$.

(iii) Now let μ be a measure with $\mu[-n,n) < \infty$. The latter means that the function $F_\mu(x)$, as defined in part (i), is finite for every $x \in \mathbb{R}$. Now take this F_μ and define, as in (ii) a (uniquely defined) measure ν_{F_μ}. Let us see that $\mu = \nu_{F_\mu}$. For this, it is enough to show equality on the sets of type $[a,b)$ (since such sets generate the Borel sets and the uniqueness theorem applies....)

If $0 \leqslant a \leqslant b$,

$$\begin{aligned}\nu_{F_\mu}[a,b) = F_\mu(b) - F_\mu(a) &= \mu[0,b) - \mu[0,a)\\ &= \mu\big([0,b) \setminus [0,a)\big)\\ &= \mu[a,b) \quad \checkmark\end{aligned}$$

If $a \leqslant b \leqslant 0$,

$$\begin{aligned}\nu_{F_\mu}[a,b) = F_\mu(b) - F_\mu(a) &= -\mu[b,0) - (-\mu[a,0))\\ &= \mu[a,0) - \mu[b,0)\\ &= \mu\big([a,0) \setminus [b,0)\big)\\ &= \mu[a,b) \quad \checkmark\end{aligned}$$

If $a \leqslant 0 \leqslant b$,

$$\begin{aligned}\nu_{F_\mu}[a,b) = F_\mu(b) - F_\mu(a) &= \mu[0,b)) - (-\mu[a,0))\\ &= \mu[a,0)) + \mu[0,b)\\ &= \mu\big([a,0) \cup [0,b)\big)\\ &= \mu[a,b) \quad \checkmark\end{aligned}$$

(iv) $F : \mathbb{R} \to \mathbb{R}$ with $F(x) = x$, since $\lambda[a,b) = b - a = F(b) - F(a)$.

(v) $F : \mathbb{R} \to \mathbb{R}$, with, say, $F(x) = \begin{cases} 0, & x \leqslant 0 \\ 1, & x > 0 \end{cases} = \mathbb{1}_{(0,\infty)}(x)$ since $\delta_0[a,b) = 0$ whenever $a, b < 0$ or $a, b > 0$. This means that F must be constant on $(-\infty, 0)$ and $(0, \infty)$ If $a \leqslant 0 < b$ we have, however, $\delta_0[a,b) = 1$ which indicates that $F(x)$ must jump by 1 at the point 0. Given the fact that F must be left-continuous, it is clear that it has, in principle, the above form. The only ambiguity is, that if $F(x)$ does the job, so does $c + F(x)$ for any constant $c \in \mathbb{R}$.

(vi) Assume that F is continuous at the point x. Then

$$\begin{aligned} \mu(\{x\}) &= \mu\left(\bigcap_{k\in\mathbb{N}} \left[x, x + \tfrac{1}{k}\right)\right) \\ &\overset{4.4}{=} \lim_{k\to\infty} \mu\left(\left[x, x + \tfrac{1}{k}\right)\right) \\ &\overset{\text{def}}{=} \lim_{k\to\infty} \left(F\left(x + \tfrac{1}{k}\right) - F(x)\right) \\ &= \lim_{k\to\infty} F\left(x + \tfrac{1}{k}\right) - F(x) \\ &\overset{(*)}{=} F(x) - F(x) = 0 \end{aligned}$$

where we used (right-)continuity of F at x in the step marked $(*)$.

Now, let conversely $\mu(\{x\}) = 0$. A similar calculation as above shows, that for *every* sequence $\epsilon_k > 0$ with $\epsilon_k \to \infty$

$$\begin{aligned} F(x+) - F(x) &= \lim_{k\to\infty} F\left(x + \epsilon_k\right) - F(x) \\ &\overset{\text{def}}{=} \lim_{k\to\infty} \mu[x, x + \epsilon_k) \\ &\overset{4.4}{=} \mu\left(\bigcap_{k\in\mathbb{N}} [x, x + \epsilon_k)\right) \\ &= \mu(\{x\}) = 0 \end{aligned}$$

which means that $F(x) = F(x+)$ ($x+$ indicates the right limit), i.e. F is right-continuous at x, hence continuous, as F is left-continuous anyway.

(vii) Then hint is indeed already the proof. Almost, that is... Let μ be some measure as specified in the problem. From part (iii) we know that the Stieltjes function $F := F_\mu$ then satisfies

$$\begin{aligned} \mu[a,b) = F(b) - F(a) &= \lambda^1[F(a), F(b)) \\ &\overset{(\#)}{=} \lambda^1(F([a,b))) \\ &\overset{(\#\#)}{=} \lambda^1 \circ F([a,b)). \end{aligned}$$

The crunching points in this argument are the steps (#) and (##).

(#) This is o.k. since F was continuous, and the intermediate value theorem for continuous functions tells us that intervals are mapped to intervals. So, no problem here, just a little thinking needed.

(##) This is more subtle. We have defined image measures *only* for inverse maps, i.e. for expressions of the type $\lambda^1 \circ G^{-1}$ where G was measurable. So our job is to see that F can be obtained in the form $F = G^{-1}$ where G is measurable. In other words, we have to invert F. The problem is that we need to understand that, if $F(x)$ is flat on some interval (a,b) inversion becomes a problem (since then F^{-1} has a jump—horizontals become verticals in inversions, as inverting is somehow the mirror-image w.r.t. the 45-degree line in the coordinate system.).

So, if there are no flat bits, then this means that F is strictly increasing, and it is clear that G exists and is even continuous there.

If we have a flat bit, let's say exactly if $x \in [a,b]$ and call $F(x) = F(a) = F(b) = C$ for those x; clearly, F^{-1} jumps at C and we must see to it that we take a version of F^{-1}, say one which makes F^{-1} left-continuous at C—note that we could assign any value from $[a,b]$ to $F^{-1}(C)$—which is accomplished by setting $F^{-1}(C) = a$. (Draw a graph to illustrate this!)

There is A canonical expression for such a 'generalized' left-continuous inverse of an increasing function (which may have jumps and flat bits—jumps of F become just flat bits in the graph of F^{-1}, think!) and this is:

$$G(y) = \inf\{x \,:\, F(x) \geqslant y\}$$

Let us check measurability:

$$\begin{aligned} y_0 \in \{G \geqslant \lambda\} &\iff G(y_0) \geqslant \lambda \\ &\overset{\text{def}}{\iff} \inf\{F \geqslant y_0\} \geqslant \lambda \\ &\overset{(\ddagger)}{\implies} F(\lambda) \leqslant y_0 \\ &\iff y_0 \in [F(\lambda), \infty). \end{aligned}$$

Since F is monotonically increasing, we find also '$\Longleftarrow$' in step $(\ddagger)$, hence

$$\{G \geqslant \lambda\} = [F(\lambda), \infty) \in \mathcal{B}(\mathbb{R})$$

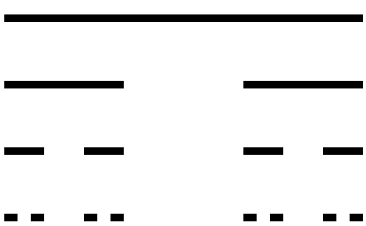
which shows that G is measurable. Even more: it shows that $G^{-1}(x) := \inf\{G \geqslant \lambda\} = F(x)$. Thus, $\lambda^1 \circ F = \lambda^1 \circ G^{-1} = \mu$ is indeed an image measure of λ^1.

(viii) We have $F(x) = F_{\delta_0}(x) = \mathbb{1}_{(0,\infty)}(x)$ and its left-continuous inverse $G(y)$ in the sense of part (vii) is given by

$$G(y) = \begin{cases} +\infty, & y > 1 \\ 0, & 0 < y \leqslant 1 \\ -\infty, & y \leqslant 0 \end{cases}.$$

This function is clearly measurable (use $\bar{\mathcal{B}}$ to accommodate $\pm\infty$) and so the claim holds in this case. Observe that in this case F is not any longer continuous but only left-continuous.

Problem 7.10 (i) We find the following picture:



$E_0 = [0,1]$

$E_1 = [0,\frac{1}{3}] \cup [\frac{2}{3},1]$

$E_2 = [0,\frac{1}{9}] \cup [\frac{2}{9},\frac{1}{3}] \cup [\frac{2}{3},\frac{7}{9}] \cup [\frac{8}{9},1]$

$E_3 = [0,\frac{1}{27}] \cup [\frac{2}{27},\frac{1}{9}] \cup [\frac{2}{9},\frac{7}{27}] \cup [\frac{8}{27},\frac{1}{3}] \cup$
$[\frac{2}{3},\frac{19}{27}] \cup [\frac{20}{27},\frac{7}{9}] \cup [\frac{8}{9},\frac{25}{27}] \cup [\frac{26}{27},1]$

(ii) Each E_n is a finite union of 2^n closed and bounded intervals. As such, E_n is itself a closed and bounded set, hence compact. The intersection of closed and bounded sets is again closed and bounded, so compact. This shows that C is compact. That C is non-empty follows from the intersection principle: if one has a nested sequence of non-empty compact sets, their intersection is not empty. (This is sometimes formulated in a somewhat stronger form and called: *finite intersection property.* The general version is then: Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of compact sets such that *each finite* sub-family has non-void intersection, then $\bigcap_n K_n \neq \emptyset$). This is an obvious generalization of the interval principle: nested non-void closed and bounded intervals have a non-void intersection.

(iii) At step n we remove open middle-third intervals of length 3^{-n}. To be precise, we partition E_{n-1} in pieces of length 3^{-n} and remove every other interval. The same effect is obtained if we partition $[0,\infty)$ in pieces of length 3^{-n} and remove every other piece. Call the taken out pieces F_n and set $E_n = E_{n-1} \setminus F_n$, i.e. we remove from E_{n-1} even pieces which were already removed in previous steps. It is clear that F_n exactly consists of sets of the form $(\frac{3k+1}{3^n},\frac{3k+2}{3^n})$, $k \in \mathbb{N}_0$ which comprises exactly 'every other' set of length 3^{-n}. Since we do this for every n, the set C is disjoint to the union of these intervals over $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$.

(iv) Since E_n consists of 2^n intervals $I_1 \uplus \ldots \uplus I_{2^n}$, each of which has length 3^{-n} (prove this by a trivial induction argument!), we get

$$\lambda(E_n) = \lambda(I_1) + \ldots + \lambda(I_{2^n}) = 2^n \cdot 3^{-n} = \left(\frac{2}{3}\right)^n$$

where we also used (somewhat pedantically) that

$$\lambda[a,b] = \lambda([a,b) \uplus \{b\}) = \lambda[a,b) + \lambda\{b\} = b - a + 0 = b - a.$$

Now using Theorem 4.4 we conclude that $\lambda(C) = \inf_n \lambda(E_n) = 0$.

(v) Fix $\epsilon > 0$ and choose n so big that $3^{-n} < \epsilon$. Then E_n consists of 2^n *disjoint* intervals of length $3^{-n} < \epsilon$ and cannot possibly contain a ball of radius ϵ. Since $C \subset E_n$, the same applies to C. Since ϵ was arbitrary, we are done. (Remark: an open ball in $\mathbb{R}$ with centre x is obviously an open interval with midpoint x, i.e. $(x-\epsilon, x+\epsilon)$.)

(vi) Fix n and let $k = 0,1,2,\dots,3^{n-1}-1$. We saw in (c) that at step n we remove the intervals F_n, i.e. the intervals of the form

$$\left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right) = \Big(0.\underbrace{*\,*\,*\dots*\,1}_{n}000\dots,\ 0.\underbrace{*\,*\,*\dots*\,2}_{n}000\dots\Big)$$

where we used the ternary representation of x. These are exactly the numbers in $[0,1]$ whose ternary expansion has a 1 at the nth digit. As $0.***\dots*1 = 0.***\dots*022222\dots$ has two representations, the left endpoint stays in. Since we do this for every step $n \in \mathbb{N}$, the claim follows.

(vii) Take $t \in C$ with ternary representation $t = 0.t_1t_2t_3\dots t_j\dots$, $t_j \in \{0,2\}$ and map it to the binary number $b = 0.\frac{t_1}{2}\frac{t_2}{2}\frac{t_3}{2}\dots\frac{t_j}{2}$ with digits $b_j = \frac{t_j}{2} \in \{0,1\}$. This gives a bijection between C and $[0,1]$, i.e. both have 'as infinitely many' points, i.e. $\#C = \#[0,1]$. Despite of that

$$\lambda(C) = 0 \neq 1 = \lambda([0,1])$$

which is, by the way, another proof for the fact that σ-additivity for the Lebesgue measure does not extend to general uncountable unions.

Problem 7.11 One direction is easy: if $f = g \circ T$ with $g : Y \to \mathbb{R}$ being measurable, we have

$$f^{-1}(\mathcal{B}(\mathbb{R})) = (g\circ T)^{-1}(\mathcal{B}(\mathbb{R})) = T^{-1}(g^{-1}(\mathcal{B}(\mathbb{R}))) \subset T^{-1}(\mathcal{A}) = \sigma(T).$$

Conversely, if f is $\sigma(T)$-measurable, then whenever $T(x) = T(x')$, we have $f(x) = f(x')$; for if not, let B be a Borel set in $\mathbb{R}$ with $f(x) \in B$ and $f(x') \notin B$. Then $f^{-1}(B) = T^{-1}(C)$ for some $C \in \mathcal{A}$, with $T(x) \in C$ but $T(x') \notin C$—which is impossible. Thus, $f = g\circ T$ for some function g from the range $T(X)$ of T. But, by assumption, $T(X) = Y$.

For any Borel set $S \subset \mathbb{R}$, $T^{-1}\circ g^{-1}(S) = f^{-1}(S) = T^{-1}(A)$ for some suitable $A \in \mathcal{A}$, so $A = g^{-1}(S)$ proving the measurability of g.

Remark. Originally, I had in mind the above solution (taken from Dudley's book [14], Theorem 4.2.8), but recently I found a much simpler solution (below) which makes the whole Remark following the statement of Problem 7.11 obsolete; moreover, it is not any longer needed to have T surjective. However, this requires that you read through Theorem 8.8 from the next chapter.

Alternative solution: One direction is easy: if $f = g\circ T$ with $g : Y \to \mathbb{R}$ being measurable, we have

$$f^{-1}(\mathcal{B}(\mathbb{R})) = (g\circ T)^{-1}(\mathcal{B}(\mathbb{R})) = T^{-1}(g^{-1}(\mathcal{B}(\mathbb{R}))) \subset T^{-1}(\mathcal{A}) = \sigma(T).$$

For the converse assume in ...

Step 1: ...first that $f = \mathbb{1}_B$ is a step function consisting of a single step. Then

$$\begin{aligned} \mathbb{1}_B \quad \sigma(T)\text{-measurable} &\iff B \in \sigma(T) \\ &\iff \exists\, A \in \mathcal{A} \,:\, B = \{T \in A\} = T^{-1}(A) \\ &\iff \mathbb{1}_B = \mathbb{1}_A \circ T \end{aligned}$$

so that $g = \mathbb{1}_A : Y \to \mathbb{R}$ does the job.

Step 2: If $f = \sum_{j=1}^N \alpha_j \mathbb{1}_{B_j}$ then, by Step 1, $g = \sum_{j=1}^N \alpha_j \mathbb{1}_{A_j}$ with the obvious notation $B_j = \{T \in A_j\}$, $A_j \in \mathcal{A}$ suitable, and $f = g \circ T$ and $g : Y \to \mathbb{R}$.

Step 3: If $f \geqslant 0$ is measurable, then we use Theorem 8.8 which says that we can approximate f as an increasing limit of $\sigma(T)$-measurable (!) elementary functions (have a look at the proof of 8.8!), say $f = \sup_j f_j$ and each f_j is of the form of the functions from Step 2. Thus, there are again functions g_j such that $f_j = g_j \circ T$ and we get

$$f = \limsup_j f_j = \limsup_j g_j \circ T = (\limsup_j g_j) \circ T$$

which means that $g := \limsup_j g_j : Y \to \mathbb{R}$ does the job.

Step 4: If f is just measurable, consider positive and negative parts $f = f^+ - f^-$ and construct, according to Step 3, $g^\pm : Y \to \mathbb{R}$ such that $g^\pm \circ T = f^\pm$. Then $g := g^+ - g^-$ does the trick.

8 Measurable functions. Solutions to Problems 8.1–8.18

Problem 8.1 We remark, first of all, that $\{u \geqslant \alpha\} = u^{-1}([x, \infty))$ and, similarly, for the other sets. Now assume that $\{u \geqslant \beta\} \in \mathcal{A}$ for all β. Then

$$\begin{aligned}
\{u > \alpha\} = u^{-1}((\alpha, \infty)) &= u^{-1}\Bigg(\bigcup_{k\in\mathbb{N}} \left[\alpha + \tfrac{1}{k}, \infty\right)\Bigg) \\
&= \bigcup_{k\in\mathbb{N}} u^{-1}\left(\left[\alpha + \tfrac{1}{k}, \infty\right)\right) \\
&= \bigcup_{k\in\mathbb{N}} \underbrace{\{u \geqslant \alpha + \tfrac{1}{k}\}}_{\text{by assumption } \in \mathcal{A}} \in \mathcal{A}
\end{aligned}$$

since $\mathcal{A}$ is a σ-algebra.

Conversely, assume that $\{u > \beta\} \in \mathcal{A}$ for all β. Then

$$\begin{aligned}
\{u \geqslant \alpha\} = u^{-1}([\alpha, \infty)) &= u^{-1}\Bigg(\bigcap_{k\in\mathbb{N}} \left(\alpha - \tfrac{1}{k}, \infty\right)\Bigg) \\
&= \bigcap_{k\in\mathbb{N}} u^{-1}\left(\left(\alpha - \tfrac{1}{k}, \infty\right)\right) \\
&= \bigcap_{k\in\mathbb{N}} \underbrace{\{u > \alpha - \tfrac{1}{k}\}}_{\text{by assumption } \in \mathcal{A}} \in \mathcal{A}.
\end{aligned}$$

since $\mathcal{A}$ is a σ-algebra. Finally, as

$$\{u > \alpha\}^c = \{u \leqslant \alpha\} \qquad \text{and} \qquad \{u \geqslant \alpha\}^c = \{u < \alpha\}$$

we have that $\{u > \alpha\} \in \mathcal{A}$ if, and only if, $\{u \leqslant \alpha\} \in \mathcal{A}$ and the same holds for the sets $\{u \geqslant \alpha\}, \{u < \alpha\}$.

Problem 8.2 Recall that $B^* \in \bar{\mathcal{B}}$ if, and only if $B^* = B \cup C$ where $B \in \mathcal{B}$ and C is any of the following sets: $\varnothing, \{-\infty\}, \{\infty\}, \{-\infty, \infty\}$. Using the fact that $\mathcal{B}$ is a σ-algebra and using this notation (that is: $\bar{\mathcal{B}}$-sets carry an asterisk *) we see

(Σ_1) Take $B = \varnothing \in \mathcal{B}$, $C = \varnothing$ to see that $\varnothing^* = \varnothing \cup \varnothing \in \bar{\mathcal{B}}$;

(Σ_2) Let $B^* \in \overline{\mathcal{B}}$. Then (complements are to be taken in $\bar{\mathcal{B}}$

$$\begin{aligned}
(B^*)^c &= (B \cup C)^c \\
&= B^c \cap C^c
\end{aligned}$$

$$
\begin{aligned}
&= (\overline{\mathbb{R}} \setminus B) \cap (\overline{\mathbb{R}} \setminus C)\\
&= (\mathbb{R} \setminus B \cup \{-\infty, +\infty\}) \cap (\overline{\mathbb{R}} \setminus C)\\
&= ((\mathbb{R} \setminus B) \cap (\overline{\mathbb{R}} \setminus C)) \cup (\{-\infty, +\infty\} \cap (\overline{\mathbb{R}} \setminus C))\\
&= (\mathbb{R} \setminus B) \cup (\{-\infty, +\infty\} \cap (\overline{\mathbb{R}} \setminus C))
\end{aligned}
$$

which is again of the type $\mathcal{B}$-set union a set of the list $\emptyset, \{-\infty\}, \{\infty\}, \{-\infty, \infty\}$, hence it is in $\bar{\mathcal{B}}$.

(Σ_3) Let $B_n^* \in \overline{\mathcal{B}}$ and $B_n^* = B_n \cup C_n$. Then

$$B^* = \bigcup_{n\in\mathbb{N}} B_n^* = \bigcup_{n\in\mathbb{N}} (B_n \cup C_n) = \bigcup_{n\in\mathbb{N}} B_n \cup \bigcup_{n\in\mathbb{N}} C_n = B \cup C$$

with $B \in \mathcal{B}$ and C from the list $\emptyset, \{-\infty\}, \{\infty\}, \{-\infty, \infty\}$, hence $B^* \in \bar{\mathcal{B}}$.

A problem is the notation $\bar{\mathcal{B}} = \mathcal{B}(\bar{\mathbb{R}})$. While the left-hand side can easily be defined by (8.5), $\mathcal{B}(\bar{\mathbb{R}})$ has a well-defined meaning as the (topological) Borel σ-algebra over the set $\bar{\mathbb{R}}$, i.e. the σ-algebra in $\bar{\mathbb{R}}$ which is defined via the open sets in $\bar{\mathbb{R}}$. To describe the open sets $(\bar{\mathbb{R}})$ of $\bar{\mathbb{R}}$ we use require, that each point $x \in U^* \in \mathcal{O}(\bar{\mathbb{R}})$ admits an open neighbourhood $B(x)$ inside U^*. If $x \neq \pm\infty$, we take $B(x)$ as the usual open ϵ-interval around x with $\epsilon > 0$ sufficiently small. If $x = \pm\infty$ we take half-lines $[-\infty, a)$ or $(b, +\infty]$ respectively with $|a|, |b|$ sufficiently large. Thus, $\mathcal{O}(\bar{\mathbb{R}})$ adds to $\mathcal{O}(\mathbb{R})$ a few extra sets and open sets are therefore of the form $U^* = U \cup C$ with $U \in \mathcal{O}(\mathbb{R})$ and C being of the form $[-\infty, a)$ or $(b, +\infty]$ or $\emptyset$ or $\bar{\mathbb{R}}$ or unions thereof.

Thus, $\mathcal{O}(\mathbb{R}) = \mathbb{R} \cap \mathcal{O}(\bar{\mathbb{R}})$ and therefore

$$\mathcal{B}(\mathbb{R}) = \mathbb{R} \cap \mathcal{B}(\bar{\mathbb{R}})$$

(this time in the proper topological sense).

Problem 8.3 (i) Notice that the indicator functions $\mathbb{1}_A$ and $\mathbb{1}_{A^c}$ are measurable. By Corollary 8.10 sums and products of measurable functions are again measurable. Since $h(x)$ can be written in the form $h(x) = \mathbb{1}_A(x)f(x) + \mathbb{1}_{A^c}(x)g(x)$, the claim follows.

(ii) The condition $f_j|_{A_j\cap A_k} = f_k|_{A_j\cap A_k}$ just guarantees that $f(x)$ is well-defined if we set $f(x) = f_j(x)$ for $x \in A_j$. Using $\bigcup_j A_j = X$ we find for $B \in \mathcal{B}(\mathbb{R})$

$$f^{-1}(B) = \bigcup_{j\in\mathbb{N}} A_j \cap f^{-1}(B) = \bigcup_{j\in\mathbb{N}} \underbrace{A_j \cap f_j^{-1}(B)}_{\in\mathcal{A}} \in \mathcal{A}.$$

An **alternative solution** would be to make the A_j's disjoint, e.g. by setting $C_1 := A_1$, $C_k := A_k \setminus (A_1 \cup \dots \cup A_{k-1})$. Then

$$f = \sum_j \mathbb{1}_{C_j} f = \sum_j \mathbb{1}_{C_j} f_j$$

and the claim follows from Corollaries 8.10 and 8.9.

Problem 8.4 Since $\mathbb{1}_B$ is $\mathcal{B}$-measurable if, and only if, $B \in \mathcal{B}$ the claim follows by taking $B \in \mathcal{B}$ such that $B \notin \mathcal{A}$ (this is possible as $\mathcal{B} \subsetneq \mathcal{A}$.

Problem 8.5 By definition, $f \in \mathcal{E}$ if it is a step-function of the form $f = \sum_{j=0}^N a_j \mathbb{1}_{A_j}$ with some $a_j \in \mathbb{R}$ and $A_j \in \mathcal{A}$. Since

$$f^+ = \sum_{\substack{0 \leqslant j \leqslant N \\ a_j \geqslant 0}} a_j \mathbb{1}_{A_j} \quad \text{and} \quad f^- = \sum_{\substack{0 \leqslant j \leqslant N \\ a_j \leqslant 0}} a_j \mathbb{1}_{A_j},$$

$f^\pm$ are again of this form and therefore simple functions.

The converse is also true since $f_f^+ - f^-$—see (8.9) or Problem 8.6—and since sums and differences of simple functions are again simple.

Problem 8.6 By definition

$$u^+(x) = \max\{u(x), 0\} \quad \text{and} \quad u^-(x) = -\min\{u(x), 0\}.$$

Now the claim follows from the elementary identities that for any two numbers $a, b \in \mathbb{R}$

$$a + 0 = \max\{a, 0\} + \min\{a, 0\} \quad \text{and} \quad |a| = \max\{a, 0\} - \min\{a, 0\}$$

which are easily verified by considering all possible cases $a \leqslant 0$ resp. $a \geqslant 0$.

Problem 8.7 Assume that $0 \leqslant u(x) \leqslant c$ for all x and some constant c. Choose $j \in \mathbb{N}$ such that $j > c$. Then the procedure used to approximate u in the proof of Theorem 8.8—see page 62, line 9 from above—guarantees that $|f_j(x) - u(x)| \leqslant 2^{-j}$ for all values of x; note that the case $u \geqslant j$ does not occur! This means that $\sup |f_j - u^+| \leqslant 2^{-j}$, i.e. we have uniform convergence.

The general case is now obtained by considering positive and negative parts $u = u^+ - u^-$ which are bounded since $u^\pm \leqslant |u| \leqslant c$.

Problem 8.8 If we show that $\{u > \alpha\}$ is an open set, it is also a Borel set, hence u is measurable.

Let us first understand what openness means: $\{u > \alpha\}$ is open means that for $x \in \{u > \alpha\}$ we find some (symmetric) neighbourhood (a 'ball') of the type $(x - h, x + h) \subset \{u > \alpha\}$. What does this mean? Obviously, that $u(y) > \alpha$ for any $y \in (x - h, x + h)$ and, in other words, $u(y) > \alpha$ whenever y is such that $|x - y| < h$. And this is the hint of how to use continuity: we use it in order to find the value of h.

u being continuous at x means that

$$\forall\, \epsilon > 0 \quad \exists\, \delta > 0 \quad \forall y : |x - y| < \delta : |u(x) - u(y)| < \epsilon.$$

Since $u(x) > \alpha$ we know that for a sufficiently small ϵ we still have $u(x) \geqslant \alpha + \epsilon$. Take this ϵ and find the corresponding δ. Then

$$u(x) - u(y) \leqslant |u(x) - u(y)| < \epsilon \qquad \forall\, |x - y| < \delta$$

and since $\alpha + \epsilon \leqslant u(x)$ we get

$$\alpha + \epsilon - u(y) < \epsilon \qquad \forall\, |x - y| < \delta$$

i.e. $u(y) > \alpha$ for y such that $|x - y| < \delta$. This means, however, that $h = \delta$ does the job.

Problem 8.9 The minimum/maximum of two numbers $a, b \in \mathbb{R}$ can be written in the form

$$\min\{a, b\} = \frac{1}{2}\big(a + b - |a - b|\big)$$
$$\max\{a, b\} = \frac{1}{2}\big(a + b + |a - b|\big)$$

which shows that we can write $\min\{x, 0\}$ and $\max\{x, 0\}$ as a combination of continuous functions. As such they are again continuous, hence measurable. Thus,

$$u^+ = \max\{u, 0\}, \qquad u^- = -\min\{u, 0\}$$

are compositions of measurable functions, hence measurable.

Problem 8.10 The f_j are step-functions where the bases of the steps are the sets A_k^j and A_j. Since they are of the form, e.g. $\big\{k2^{-j} \leqslant u < (k+1)2^{-j}\big\} = \big\{k2^{-j} \leqslant u\big\} \cap \big\{u < (k+1)2^{-j}\big\}$, it is clear that they are not only in $\mathcal{A}$ but in $\sigma(u)$.

Problem 8.11 **Corollary 8.11** If $u^{\pm}$ are measurable, it is clear that $u = u^+ - u^-$ is measurable since differences of measurable functions are measurable.

(For the converse we could use the previous Problem 8.10, but we give an alternative proof...) Conversely, let u be measurable. Then $s_n \uparrow u$ (this is short for: $\lim_{n\to\infty} s_n(x) = u(x)$ and this is an increasing limit) for some sequence of simple functions s_n. Now it is clear that $s_n^+ \uparrow u^+$, and s_n^+ is simple, i.e. u^+ is measurable. As $u = u^+ - u^-$ we conclude that $u^- = u^+ - u$ is again measurable as difference of two measurable functions. (Notice that in no case '$\infty - \infty$' can occur!)

Corollary 8.12 This is trivial if the difference $u - v$ is defined. In this case it is measurable as difference of measurable functions, so

$$\{u < v\} = \{0 < u - v\}$$

etc. is measurable.

Let us be a bit more careful and consider the case where we could encounter expressions of the type '$\infty - \infty$'. Since $s_n \uparrow u$ for simple functions (they are always $\mathbb{R}$-valued...) we get

$$\{u \leqslant v\} = \{\sup_n s_n \leqslant u\} \overset{(*)}{=} \bigcap_n \{s_n \leqslant u\} = \bigcap_n \{0 \leqslant u - s_n\}$$

and the latter is a union of measurable sets, hence measurable. Now $\{u < v\} = \{u \geqslant v\}^c$ and we get measurability after switching the roles of u and v. Finally $\{u = v\} = \{u \leqslant v\} \cap \{u \geqslant v\}$ and $\{u \neq v\} = \{u = v\}^c$.

Let me stress the importance of '$\leqslant$' in $(*)$ above: we use here

$$\begin{aligned} x \in \{\sup_n s_n \leqslant u\} &\iff \sup_n s_n(x) \leqslant u(x) \\ &\overset{(**)}{\iff} s_n(x) \leqslant u(x) \quad \forall\, n \\ &\iff x \in \bigcap \{s_n \leqslant u\} \end{aligned}$$

and this would be incorrect if we had had '<', since the argument would break down at $(**)$ (only one implication would be valid: '$\Longrightarrow$').

Problem 8.12 If u is differentiable, it is continuous, hence measurable. Moreover, since u' exists, we can write it in the form

$$u'(x) = \lim_{k\to\infty} \frac{u\big(x+\frac{1}{k}\big) - u(x)}{\frac{1}{k}}$$

i.e. as limit of measurable functions. Thus, u' is also measurable.

Problem 8.13 It is sometimes necessary to distinguish between domain and range. We use the subscript x to signal the domain, the subscript y for the range.

(i) Since $f : \mathbb{R}_x \to \mathbb{R}_y$ is $f(x) = x$, the inverse function is clearly $f^{-1}(y) = y$. So if we take any Borel set $B \in \mathcal{B}(\mathbb{R}_y)$ we get $B = f^{-1}(B) \subset \mathbb{R}_x$. Since, as we have seen, $\sigma(f) = f^{-1}(\mathcal{B}(\mathbb{R}_y))$, the above argument shows that $f^{-1}(\mathcal{B}(\mathbb{R}_y)) = \mathcal{B}(\mathbb{R}_x)$, hence $\sigma(f) = \mathcal{B}(\mathbb{R}_x)$.

(ii) The inverse map of $g(x) = x^2$ is multi-valued, i.e. if $y = x^2$, then $y = \pm\sqrt{x}$. So $g^{-1} : [0,\infty) \to \mathbb{R}$, $g^{-1}(y) = \pm\sqrt{y}$. Let us take some $B \in \mathcal{B}(\mathbb{R}_y)$. Since g^{-1} is only defined for positive numbers (squares yield positive numbers only!) we have that $g^{-1}(B) = g^{-1}(B \cap [0,\infty)) = \sqrt{B\cap[0,\infty)} \cup (-\sqrt{B\cap[0,\infty)})$ (where we used the obvious notation $\sqrt{A} = \{\sqrt{a} : a \in A\}$ and $-A = \{-a : a \in A\}$ whenever A is a set). This shows that

$$\begin{aligned} \sigma(g) &= \{\sqrt{B} \cup (-\sqrt{B}) \;:\; B \in \mathcal{B}, B \subset [0,\infty)\} \\ &= \{\sqrt{B} \cup (-\sqrt{B}) \;:\; B \in [0,\infty) \cap \mathcal{B}\} \end{aligned}$$

where we used the notation of trace σ-algebras in the latter identity.

(It is an instructive exercise to check that $\sigma(g)$ is indeed a σ-algebra. This is, of course, clear from the general theory since $\sigma(g) = g^{-1}([0,\infty) \cap \mathcal{B})$, i.e. it is the pre-image of the trace σ-algebra and pre-images of σ-algebras are always σ-algebras.

(iii) A very similar calculation as in part (ii) shows that

$$\begin{aligned} \sigma(h) &= \{B \cup (-B) \;:\; B \in \mathcal{B}, B \subset [0,\infty)\} \\ &= \{B \cup (-B) \;:\; B \in [0,\infty) \cap \mathcal{B}\}. \end{aligned}$$

(iv) As warm-up we follow the hint. The set $\{(x,y) : x+y=\alpha\}$ is the line $y=\alpha-x$ in the x-y-plane, i.e. a line with slope -1 and shift α. So $\{(x,y) : x+y\geqslant\alpha\}$ would be the points above this line and $\{(x,y) : \beta\geqslant x+y\geqslant\alpha\}=\{(x,y) : x+y\in[\alpha,\beta]\}$ would be the points in the strip which has the lines $y=\alpha-x$ and $y=\beta-x$ as boundaries.

More general, take a Borel set $B\in\mathcal{B}(\mathbb{R})$ and observe that

$$F^{-1}(B)=\{(x,y) : x+y\in B\}.$$

This set is, in an abuse of notation, $y=B-x$, i.e. these are all lines with slope -1 (135 degrees) and *every possible shift from the set* B—it gives a kind of stripe-pattern. To sum up:

$$\sigma(F)=\{\text{all 135-degree diagonal stripes in } \mathbb{R}^2 \text{ with 'base' } B\in\mathcal{B}(\mathbb{R})\}.$$

(v) Again follow the hint to see that $\{(x,y) : x^2+y^2=r\}$ is a circle, radius r, centre $(0,0)$. So $\{(x,y) : x^2+y^2\leqslant r\}$ is the solid disk, radius r, centre $(0,0)$ and $\{(x,y) : R\geqslant x^2+y^2\geqslant r\}=\{(x,y) : x^2+y^2\in[r,R]\}$ is the annulus with exterior radius R and interior radius r about $(0,0)$.

More general, take a Borel set $B\subset[0,\infty), B\in\mathcal{B}(\mathbb{R})$, i.e. $B\in[0,\infty)\cap\mathcal{B}(\mathbb{R})$ (negative radii don't make sense!) and observe that the set $\{(x,y) : x^2+y^2\in B\}$ gives a ring-pattern which is 'supported' by the set B (i.e. we take all circles passing through B...). To sum up:

$$\sigma(G)=\{\text{a set consists of all circles in } \mathbb{R}^2 \text{ about } (0,0)$$
$$\text{passing through } B\in[0,\infty)\cap B(\mathbb{R})\}.$$

Problem 8.14 Assume first that u is injective. This means that every point in the range $u(\mathbb{R})$ comes exactly from one uniquely defined $x\in\mathbb{R}$. This can be expressed by saying that $\{x\}=u^{-1}(\{u(x)\})$ — but the singleton $\{u(x)\}$ is a Borel set in the range, so $\{x\}\in\sigma(u)$ as $\sigma(u)=u^{-1}(u(\mathbb{R})\cap\mathcal{B})$.

Conversely, assume that for each x we have $\{x\}\in\sigma(u)$. Fix an x_0 and call $u(x_0)=\alpha$. Since u is measurable, the set $\{u=\alpha\}=\{x : u(x)=\alpha\}$ is measurable and, clearly, $\{x_0\}\subset\{u=\alpha\}$. But if we had another $x_0\neq x_1\in\{u=\alpha\}$ this would mean that we could never 'produce' $\{x_0\}$ on its own as a pre-image of some set, but we must be able to do so as $\{x_0\}\in\sigma(u)$, by assumption. Thus, $x_1=x_0$. To sum up, we have shown that $\{u=\alpha\}$ consists of one point only, i.e. we have shown that $u(x_0)=u(x_1)$ implies $x_0=x_1$ which is just injectivity.

Problem 8.15 Clearly $u:\mathbb{R}\to[0,\infty)$. So let's take $I=(a,b)\subset[0,\infty)$. Then $u^{-1}((a,b))=(-b,-a)\cup(a,b)$. This shows that for $\mu:=\lambda\circ u^{-1}$

$$\mu(a,b)=\lambda\circ u^{-1}((a,b))=\lambda\big((-b,-a)\cup(a,b)\big)=\lambda(-b,-a)+\lambda(a,b)$$

$$= (-a - (-b)) + (b - a) = 2(b - a) = 2\lambda((a, b)).$$

This shows that $\mu = 2\lambda$ if we allow only intervals from $[0, \infty)$, i.e.

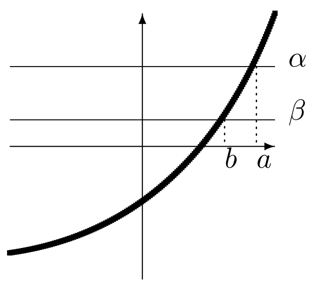
$$\mu(I) = 2\lambda\big(I \cap [0, \infty)\big) \quad \text{for any interval } I \subset \mathbb{R}.$$

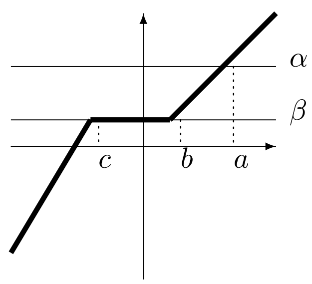
Since a measure on the Borel sets is completely described by (either: open or closed or half-open or half-closed) intervals (the intervals generate the Borel sets!), we can invoke the uniqueness theorem to guarantee that the above equality holds for all Borel sets.

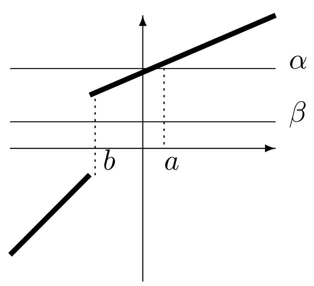
Problem 8.16
- clear, since $u(x - 2)$ is a combination of the measurable shift τ_2 and the measurable function u.
- this is trivial since $u \mapsto e^u$ is a continuous function, as such it is measurable and combinations of measurable functions are again measurable.
- this is trivial since $u \mapsto \sin(u + 8)$ is a continuous function, as such it is measurable and combinations of measurable functions are again measurable.
- iterate Problem 8.12
- obviously, $\operatorname{sgn} x = (-1) \cdot \mathbb{1}_{(-\infty,0)}(x) + 0 \cdot \mathbb{1}_{\{0\}}(x) + 1 \cdot \mathbb{1}_{(0,\infty)}(x)$, i.e. a measurable function. Using the first example, we see now that $\operatorname{sgn} u(x - 7)$ is a combination of three measurable functions.

Problem 8.17 Let $A \subset \mathbb{R}$ be such that $A \notin \mathcal{B}$. Then it is clear that $u(x) = \mathbb{1}_A(x) - \mathbb{1}_{A^c}(x)$ is NOT measurable (take, e.g. $A = \{f = 1\}$ which should be measurable for measurable functions), but clearly, $|f(x)| = 1$ and as constant function this IS measurable.

Problem 8.18 We want to show that the sets $\{u \leqslant \alpha\}$ are Borel sets. We will even show that they are intervals, hence Borel sets. Imagine the graph of an increasing function and the line $y = \alpha$ cutting through. Essentially we have three scenarios: the cut happens at a point where (a) u is continuous and strictly increasing or (b) u is flat or (c) u jumps—i.e. has a gap; these three cases are shown in the following pictures:







From the three pictures it is clear that we get in any case an interval for the sub-level sets $\{u \leqslant \gamma\}$ where γ is some level (in the pic's $\gamma = \alpha$ or $= \beta$), you can read off the intervals on the abscissa where the dotted lines cross the abscissa.

Now let's look at the additional conditions: First the intuition: From the first picture, the continuous and strictly increasing case, it is clear that we can produce any interval $(-\infty, b]$ to $(-\infty, a]$ by looking at $\{u \leqslant \beta\}$ to $\{u \leqslant \alpha\}$ my moving up the β-line to level α. The point is here that we get all intervals, so we get a generator of the Borel sets, so we should get all Borel sets.

The second picture is bad: the level set $\{u \leqslant \beta\}$ is $(-\infty, b]$ and all level sets below will only come up to the point $(-\infty, c]$, so there is no chance to get any set contained in (c, b), i.e. we cannot get all Borel sets.

The third picture is good again, because the vertical jump does not hurt. The only 'problem' is whether $\{u \leqslant \beta\}$ is $(-\infty, b]$ or $(-\infty, b)$ which essentially depends on the property of the graph whether $u(b) = \beta$ or not, but this is not so relevant here, we just must make sure that we can get more or less all intervals. The reason, really, is that jumps as we described them here can only happen countably often, so this problem occurs only countably often, and we can overcome it therefore.

So the point is: we must disallow flat bits, i.e. $\sigma(u)$ is the Borel σ-algebra if, and only, if u is strictly increasing, i.e. if, and only if, u is injective. (Note that this would have been clear already from Problem 8.14, but our approach here is much more intuitive.)

9 Integration of positive functions. Solutions to Problems 9.1–9.12

Problem 9.1 We know that for any two simple functions $f, g \in \mathcal{E}_+$ we have $I_\mu(f+g) = I_\mu(f) + I_\mu(g)$ (=additivity), and this is easily extended to finitely many, say, m different positive simple functions. Observe now that each $\xi_j \mathbb{1}_{A_j}$ is a positive simple function, hence

$$I_\mu\left(\sum_{j=1}^m \xi_j \mathbb{1}_{A_j}\right) = \sum_{j=1}^m I_\mu\left(\xi_j \mathbb{1}_{A_j}\right) = \sum_{j=1}^m \xi_j I_\mu\left(\mathbb{1}_{A_j}\right) = \sum_{j=1}^m \xi_j \mu\left(A_j\right).$$

Put in other words: we have used the linearity of I_μ.

Problem 9.2 We check Properties 9.8(i)–(iv).

(i) This follows from Properties 9.3 and Lemme 9.5 since $\int \mathbb{1}_A \, d\mu = I_\mu(\mathbb{1}_A) = \mu(A)$.

(ii) This follows again from Properties 9.3 and Corollary 9.7 since for $u_n \in \mathcal{E}_+$ with $u = \sup_n u_n$ (note: the sup's are increasing limits!) we have

$$\begin{aligned}\int \alpha u \, d\mu = \int \alpha \sup_n u_n \, d\mu &= \sup_n I_\mu(\alpha u_n)\\ &= \sup_n \alpha I_\mu(u_n)\\ &= \alpha \sup_n I_\mu(u_n)\\ &= \alpha \int u \, d\mu.\end{aligned}$$

(iii) This follows again from Properties 9.3 and Corollary 9.7 since for $u_n, v_n \in \mathcal{E}_+$ with $u = \sup_n u_n, v = \sup_n v_n$ (note: the sup's are increasing limits!) we have

$$\begin{aligned}\int (u+v) \, d\mu = \int \lim_{n\to\infty} (u_n + v_n) \, d\mu &= \lim_{n\to\infty} I_\mu(u_n + v_n)\\ &= \lim_{n\to\infty} \left(I_\mu(u_n) + I_\mu(v_n)\right)\\ &= \lim_{n\to\infty} I_\mu(u_n) + \lim_{n\to\infty} I_\mu(v_n)\\ &= \int u \, d\mu + \int v \, d\mu.\end{aligned}$$

(iv) This was shown in step 1 of the proof of the Beppo Levi theorem 9.6

Problem 9.3 Consider on the space $([-1,0], \lambda)$, $\lambda(dx) = dx$ is Lebesgue measure on $[0,1]$, the sequence of 'tent-type' functions

$$f_k(x) = \begin{cases} 0, & -1 \leqslant x \leqslant -\frac{1}{k}, \\ k^3\left(x + \frac{1}{k}\right), & -\frac{1}{k} \leqslant x \leqslant 0, \end{cases} \qquad (k \in \mathbb{N}),$$

(draw a picture!). These are clearly monotonically increasing functions but, as a sequence, we do not have $f_k(x) \leqslant f_{k+1}(x)$ for *every* x! Note also that each function is integrable (with integral $\frac{1}{2}k$) but the pointwise limit is not integrable.

Problem 9.4 Following the hint we set $s_m = u_1 + u_2 + \ldots + u_m$. As a finite sum of positive measurable functions this is again positive and measurable. Moreover, s_m increases to $s = \sum_{j=1}^\infty u_j$ as $m \to \infty$. Using the additivity of the integral (9.8 (iii)) and the Beppo Levi theorem 9.6 we get

$$\begin{aligned}\int \sum_{j=1}^\infty u_j\,d\mu = \int \sup_m s_m\,d\mu = \sup_m \int s_m\,d\mu &= \sup_m \int (u_1 + \ldots + u_m)\,d\mu \\ &= \sup_m \sum_{j=1}^m \int u_j\,d\mu \\ &= \sum_{j=1}^\infty \int u_j\,d\mu.\end{aligned}$$

Conversely, assume that 9.9 is true. We want to deduce from it the validity of Beppo Levi's theorem 9.6. So let $(w_j)_{j\in\mathbb{N}}$ be an increasing sequence of measurable functions with limit $w = \sup_j w$. For ease of notation we set $w_0 \equiv 0$. Then we can write each w_j as a partial sum

$$w_j = (w_j - w_{j-1}) + \cdots + (w_1 - w_0)$$

of positive measurable summands of the form $u_k := w_k - w_{k-1}$. Thus,

$$w_m = \sum_{k=1}^m u_k \quad \text{and} \quad w = \sum_{k=1}^\infty u_k$$

and, using the additivity of the integral,

$$\int w\,d\mu \overset{9.9}{=} \sum_{k=1}^\infty \int u_k\,d\mu = \sup_m \int \sum_{k=1}^m u_k\,d\mu = \sup_m \int w_m\,d\mu.$$

Problem 9.5 Set $\nu(A) := \int \mathbb{1}_A u\,d\mu$. Then ν is a $[0,\infty]$-valued set-function defined for $A \in \mathcal{A}$.

(M_1) Since $\mathbb{1}_\emptyset \equiv 0$ we have clearly $\nu(\emptyset) = \int 0 \cdot u\,d\mu = 0$.

(M_1) Let $A = \bigcup_{j\in\mathbb{N}} A_j$ a disjoint union of sets $A_j \in \mathcal{A}$. Then

$$\sum_{j=1}^\infty \mathbb{1}_{A_j} = \mathbb{1}_A$$

and we get from Corollary 9.9

$$\begin{aligned}\nu(A) = \int \left(\sum_{j=1}^\infty \mathbb{1}_{A_j}\right) \cdot u\,d\mu = \int \sum_{j=1}^\infty \big(\mathbb{1}_{A_j} \cdot u\big)\,d\mu &= \sum_{j=1}^\infty \int \mathbb{1}_{A_j} \cdot u\,d\mu \\ &= \sum_{j=1}^\infty \nu(A_j).\end{aligned}$$

Problem 9.6 This is actually trivial: since our σ-algebra is $\mathcal{P}(\mathbb{N})$, all subsets of $\mathbb{N}$ are measurable. Now the sub-level sets $\{u \leqslant \alpha\} = \{k \in \mathbb{N} : u(k) \leqslant \alpha\}$ are always $\subset \mathbb{N}$ and as such they are $\in \mathcal{P}(\mathbb{N})$, hence u is always measurable.

Problem 9.7 We have seen in Problem 4.6 that μ is indeed a measure. We follow the instructions. First, for $A \in \mathcal{A}$ we get

$$\int \mathbb{1}_A \, d\mu = \mu(A) = \sum_{j\in\mathbb{N}} \mu_j(A) = \sum_{j\in\mathbb{N}} \int \mathbb{1}_A \, d\mu_j.$$

By the linearity of the integral, this easily extends to functions of the form $\alpha \mathbb{1}_A + \beta \mathbb{1}_B$ where $A, B \in \mathcal{A}$ and $\alpha, \beta \geqslant 0$:

$$\begin{aligned}\int (\alpha \mathbb{1}_A + \beta \mathbb{1}_B) \, d\mu &= \alpha \int \mathbb{1}_A \, d\mu + \beta \int \mathbb{1}_B \, d\mu \\ &= \alpha \sum_{j\in\mathbb{N}} \int \mathbb{1}_A \, d\mu_j + \beta \sum_{j\in\mathbb{N}} \int \mathbb{1}_B \, d\mu_j \\ &= \sum_{j\in\mathbb{N}} \int (\alpha \mathbb{1}_A + \beta \mathbb{1}_B) \, d\mu_j\end{aligned}$$

and this extends obviously to simple functions which are finite sums of the above type.

$$\int f \, d\mu = \sum_{j\in\mathbb{N}} \int f \, d\mu_j \qquad \forall f \in \mathcal{E}_+.$$

Finally, take $u \in \mathcal{M}_+$ and take an approximating sequence $u_n \in \mathcal{E}_+$ with $\sup_n u_n = u$. Then we get by Beppo Levi (indicated by an asterisk $*$)

$$\begin{aligned}\int u \, d\mu &\overset{*}{=} \sup_n \int u_n \, d\mu = \sup_n \sum_{j=1}^{\infty} \int u_n \, d\mu_j \\ &= \sup_n \sup_m \sum_{j=1}^{m} \int u_n \, d\mu_j \\ &= \sup_m \sup_n \sum_{j=1}^{m} \int u_n \, d\mu_j \\ &= \sup_m \lim_n \sum_{j=1}^{m} \int u_n \, d\mu_j \\ &= \sup_m \sum_{j=1}^{m} \lim_n \int u_n \, d\mu_j \\ &\overset{*}{=} \sup_m \sum_{j=1}^{m} \int \lim_n u_n \, d\mu_j \\ &= \sum_{j=1}^{\infty} \int u \, d\mu_j\end{aligned}$$

where we repeatedly used that all sup's are increasing limits and that we may swap any two sup's (this was the hint to the hint to Problem 4.6.)

Problem 9.8 Set $w_j := u - u_j$. Then the w_j are a sequence of positive measurable functions. By Fatou's lemma we get

$$\begin{aligned}\int \liminf_j w_j \, d\mu &\leqslant \liminf_j \int w_j \, d\mu \\ &= \liminf_j \left(\int u \, d\mu - \int u_j \, d\mu \right) \\ &= \int u \, d\mu - \limsup_j \int u_j \, d\mu\end{aligned}$$

(see, e.g. the rules for lim inf and lim sup in Appendix A). Thus,

$$\begin{aligned}\int u \, d\mu - \limsup_j \int u_j \, d\mu \geqslant \int \liminf_j w_j \, d\mu \\ &= \int \liminf_j (u - u_j) \, d\mu \\ &= \int \left(u - \limsup_j u_j\right) d\mu\end{aligned}$$

and the claim follows by subtracting the *finite* value $\int u \, d\mu$ on both sides.

Remark. The uniform domination of u_j by an integrable function u is really important. Have a look at the following situation: $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, $\lambda(dx) = dx$ denotes Lebesgue measure, and consider the positive measurable functions $u_j(x) = \mathbb{1}_{[j,2j]}(x)$. Then $\limsup_j u_j(x) = 0$ but $\limsup_j \int u_j \, d\lambda = \limsup_j j = \infty \neq \int 0 \, d\lambda$.

Problem 9.9 (i) Have a look at Appendix A, Lemma A.2.

(ii) You have two possibilities: the set-theoretic version:

$$\begin{aligned}\mu\left(\liminf_j A_j\right) &= \mu\left(\bigcup_k \bigcap_{j \geqslant k} A_j\right) \\ &\stackrel{*}{=} \sup_k \underbrace{\mu\left(\bigcap_{j \geqslant k} A_j\right)}_{\substack{\leqslant \mu(A_j) \ \forall j \geqslant k \\ \text{hence, } \leqslant \inf_{j \geqslant k} \mu(A_j)}} \\ &\leqslant \sup_k \inf_{j \geqslant k} \mu(A_j) \\ &= \liminf_j \mu(A_j)\end{aligned}$$

which uses at the point $*$ the continuity of measures, Theorem 4.4.

The *alternative* would be (i) combined with Fatou's lemma:

$$\begin{aligned}\mu\left(\liminf_j A_j\right) &= \int \mathbb{1}_{\liminf_j A_j} \, d\mu \\ &= \int \liminf_j \mathbb{1}_{A_j} \, d\mu \\ &\leqslant \liminf_j \int \mathbb{1}_{A_j} \, d\mu\end{aligned}$$

(iii) Again, you have two possibilities: the set-theoretic version:

$$\begin{aligned}\mu\big(\limsup_j A_j\big) &= \mu\Big(\bigcap_k \bigcup_{j\geqslant k} A_j\Big)\\ &\overset{\#}{=} \inf_k \underbrace{\mu\Big(\bigcup_{j\geqslant k} A_j\Big)}_{\substack{\geqslant \mu(A_j)\ \forall\, j\geqslant k\\ \text{hence, } \geqslant \sup_{j\geqslant k}\mu(A_j)}}\\ &\geqslant \inf_k \sup_{j\geqslant k} \mu(A_j)\\ &= \limsup_j \mu(A_j)\end{aligned}$$

which uses at the point # the continuity of measures, Theorem 4.4. This step uses the finiteness of μ.

The *alternative* would be (i) combined with the reversed Fatou lemma of Problem 9.8:

$$\begin{aligned}\mu\big(\limsup_j A_j\big) &= \int \mathbb{1}_{\limsup_j A_j}\, d\mu\\ &= \int \limsup_j \mathbb{1}_{A_j}\, d\mu\\ &\geqslant \limsup_j \int \mathbb{1}_{A_j}\, d\mu\end{aligned}$$

(iv) Take the example in the remark to the solution for Problem 9.8. We will discuss it here in its set-theoretic form: take $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ with λ denoting Lebesgue measure $\lambda(dx) = dx$. Put $A_j = [j, 2j] \in \mathcal{B}(\mathbb{R})$. Then

$$\limsup_j A_j = \bigcap_k \bigcup_{j\geqslant k} [j, 2j] = \bigcap_k [k, \infty) = \varnothing$$

But $0 = \lambda(\varnothing) \geqslant \limsup_j \lambda(A_j) = \limsup_j j = \infty$ is a contradiction. (The problem is that $\lambda[k, \infty) = \infty$!)

Problem 9.10 We use the fact that, because of disjointness,

$$1 = \mathbb{1}_X = \sum_{j=1}^{\infty} \mathbb{1}_{A_j}$$

so that, because of Corollary 9.9,

$$\begin{aligned}\int u\, d\mu = \int \Big(\sum_{j=1}^{\infty} \mathbb{1}_{A_j}\Big) \cdot u\, d\mu &= \int \sum_{j=1}^{\infty} \big(\mathbb{1}_{A_j} \cdot u\big)\, d\mu\\ &= \sum_{j=1}^{\infty} \int \mathbb{1}_{A_j} \cdot u\, d\mu.\end{aligned}$$

Assume now that $(X, \mathcal{A}, \mu)$ is σ-finite with an exhausting sequence of sets $(B_j)_j \subset \mathcal{A}$ such that $B_j \uparrow X$ and $\mu(B_j) < \infty$. Then we make the B_j's pairwise disjoint by setting

$$A_1 := B_1, \qquad A_k := B_k \setminus (B_1 \cup \dots \cup B_{k-1}) = B_k \setminus B_{k-1}.$$

Now take any sequence $(a_k)_k \subset (0,\infty)$ with $\sum_k a_k\mu(A_k) < \infty$—e.g. $a_k := 2^{-k}/(\mu(A_k)+1)$—and put

$$w(x) := \sum_{j=1}^{\infty} a_k \mathbb{1}_{A_k}.$$

Then w is integrable and, obviously, $w(x) > 0$ everywhere.

Problem 9.11 (i) We check $(M_1), (M_2)$. Using the fact that $N(x,\cdot)$ is a measure, we find

$$\mu N(\varnothing) = \int N(x,\varnothing)\,\mu(dx) = \int 0\,\mu(dx) = 0.$$

Further, let $(A_j)_{j\in\mathbb{N}} \subset \mathcal{A}$ be a sequence of disjoint sets and set $A = \dot\bigcup_j A_j$. Then

$$\begin{aligned}\mu N(A) = \int N\Big(x, \dot{\textstyle\bigcup}_j A_j\Big)\,\mu(dx) &= \int \sum_j N(x,A_j)\,\mu(dx)\\ &\overset{9.9}{=} \sum_j \int N(x,A_j)\,\mu(dx)\\ &= \sum_j \mu N(A_j).\end{aligned}$$

(ii) We have for $A,B \in \mathcal{A}$ and $\alpha,\beta \geqslant 0$,

$$\begin{aligned}N(\alpha\mathbb{1}_A + \beta\mathbb{1}_B)(x) &= \int \big(\alpha\mathbb{1}_A(y) + \beta\mathbb{1}_B(y)\big)\,N(x,dy)\\ &= \alpha\int \mathbb{1}_A(y)\,N(x,dy) + \beta\int \mathbb{1}_B(y)\,N(x,dy)\\ &= \alpha N\mathbb{1}_A(x) + \beta N\mathbb{1}_B(x).\end{aligned}$$

Thus $N(f+g)(x) = Nf(x) + Ng(x)$ for positive simple $f,g \in \mathcal{E}^+(\mathcal{A})$. Moreover, since by Beppo Levi (marked by an asterisk $*$) for an increasing sequence $f_k \uparrow u$

$$\begin{aligned}\sup_k Nf_k(x) = \sup_k \int f_k(y)\,N(x,dy) &\overset{*}{=} \int \sup_k f_k(y)\,N(x,dy)\\ &= \int u(y)\,N(x,dy)\\ &= Nu(x)\end{aligned}$$

and since the sup is actually an increasing limit, we see for positive measurable $u,v \in \mathcal{M}^+(\mathcal{A})$ and the corresponding increasing approximations via positive simple functions f_k, g_k:

$$\begin{aligned}N(u+v)(x) &= \sup_k N(f_k+g_k)(x)\\ &= \sup_k Nf_k(x) + \sup_k Ng_k(x)\\ &= Nu(x) + Nv(x).\end{aligned}$$

Moreover, $x \mapsto N\mathbb{1}_A(x) = N(x,A)$ is a measurable function, thus $Nf(x)$ is a measurable function for all simple $f \in \mathcal{E}^+(\mathcal{A})$ and, by Beppo Levi (see above) $Nu(x)$, $u \in \mathcal{M}^+(\mathcal{A})$, is for every x an increasing limit of measurable functions $Nf_k(x)$. Therefore, $Nu \in \mathcal{M}^+(\mathcal{A})$.

(iii) If $u = \mathbb{1}_A$, $A \in \mathcal{A}$, we have

$$\int \mathbb{1}_A(y)\,\mu N(dy) = \mu N(A) = \int N(x,A)\,\mu(dx)$$
$$= \int N\mathbb{1}_A(x)\,\mu(dx).$$

By linearity this carries over to $f \in \mathcal{E}^+(\mathcal{A})$ and, by a Beppo-Levi argument, to $u \in \mathcal{M}^+(\mathcal{A})$.

Problem 9.12 Put

$$\nu(A) := \int u \cdot \mathbb{1}_{A_\sigma^+}\,d\mu + \int (1-u)\cdot \mathbb{1}_{A_\sigma^-}\,d\mu.$$

If A is symmetric w.r.t. the origin, $A^+ = -A^-$ and $A_\sigma^\pm = A$. Therefore,

$$\nu(A) = \int u \cdot \mathbb{1}_A\,d\mu + \int (1-u)\cdot \mathbb{1}_A\,d\mu = \int \mathbb{1}_A\,d\mu = \mu(A).$$

This means that ν extends μ. It also shows that $\nu(\emptyset) = 0$. Since ν is defined for all sets from $\mathcal{B}(\mathbb{R})$ and since ν has values in $[0,\infty]$, it is enough to check σ-additivity.

For this, let $(A_j)_j \subset \mathcal{B}(\mathbb{R})$ be a sequence of pairwise disjoint sets. From the definitions it is clear that the sets $(A_j)_\sigma^\pm$ are again pairwise disjoint and that $\dot\bigcup_j (A_j)_\sigma^\pm = \left(\dot\bigcup_j A_j\right)_\sigma^\pm$. Since each of the set-functions

$$B \mapsto \int u \cdot \mathbb{1}_B\,d\mu, \qquad C \mapsto \int (1-u)\cdot \mathbb{1}_C\,d\mu$$

is σ-additive, it is clear that their sum ν will be σ-additive, too.

The obvious non-uniqueness of the extension does not contradict the uniqueness theorem for extensions, since Σ does not generate $\mathcal{B}(\mathbb{R})$!

10 Integrals of measurable functions and null sets. Solutions to Problems 10.1–10.16

Problem 10.1 Let u, v be integrable functions and $a, b \in \mathbb{R}$. Assume that either u, v are real-valued or that $au + bv$ makes sense (i.e. avoiding the case '$\infty - \infty$'). Then we have

$$|au + bv| \leqslant |au| + |bv| = |a| \cdot |u| + |b| \cdot |v| \leqslant K(|u| + |v|)$$

with $K = \max\{|a|, |b|\}$. Since the RHS is integrable (because of Theorem 10.3 and Properties 9.8) we have that $au + bv$ is integrable by Theorem 10.3. So we get from Theorem 10.4 that

$$\int (au + bv)\, d\mu = \int au\, d\mu + \int bv\, d\mu = a \int u\, d\mu + b \int v\, d\mu$$

and this is what was claimed.

Problem 10.2 We follow the hint and show first that $u(x) := x^{-1/2}$, $0 < x < 1$, is Lebesgue integrable. The idea here is to construct a sequence of simple functions approximating u from below. Define

$$u_n(x) := \begin{cases} 0, & \text{if } x \in (0, \frac{1}{n}) \\ u(\frac{j+1}{n}), & \text{if } x \in [\frac{j}{n}, \frac{j+1}{n}), \quad j = 1, \dots n-1 \end{cases}$$

$$\iff \qquad u_n = \sum_{j=1}^{n-1} u(\tfrac{j+1}{n}) \mathbb{1}_{\left[\frac{j}{n}, \frac{j+1}{n}\right)}$$

which is clearly a simple function. Also $u_n \leqslant u$ and $\lim_{n\to\infty} u_n(x) = \sup_n u_n(x) = u(x)$ for all x.

Since $P(A)$ is just $\lambda(A \cap (0,1))$, the integral of u_n is given by

$$\begin{aligned}\int u_n\, dP = I_P(u_n) &= \sum_{j=1}^{n-1} u(\tfrac{j+1}{n}) \lambda[\tfrac{j}{n}, \tfrac{j+1}{n}) \\ &= \sum_{j=1}^{n-1} \sqrt{\tfrac{j+1}{n}} \cdot \tfrac{1}{n} \\ &\leqslant \sum_{j=1}^{n-1} \tfrac{1}{n} \leqslant 1\end{aligned}$$

and is thus finite, even uniformly in n! So, Beppo Levi's theorem tells us that

$$\int u\, dP = \sup_n \int u_n\, dP \leqslant \sup_n 1 = 1 < \infty$$

showing integrability.

Now u is clearly not bounded but integrable.

Problem 10.3 True, we can change an integrable function on a null set, even by setting it to the value $+\infty$ or $-\infty$ *on the null set*. This is just the assertion of Theorem 10.9 and its Corollaries 10.10, 10.11.

Problem 10.4 We have seen that a single point is a Lebesgue null set: $\{x\} \in \mathcal{B}(\mathbb{R})$ for all $x \in \mathbb{R}$ and $\lambda(\{x\}) = 0$, see e.g. Problems 4.11 and 6.4. If N is countable, we know that $N = \{x_j : j \in \mathbb{N}\} = \bigcup_{j\in\mathbb{N}}\{x_j\}$ and by the σ-additivity of measures

$$\lambda(N) = \lambda\left(\bigcup_{j\in\mathbb{N}}\{x_j\}\right) = \sum_{j\in\mathbb{N}}\lambda(\{x_j\}) = \sum_{j\in\mathbb{N}} 0 = 0.$$

The Cantor set C from Problem 7.10 is, as we have seen, uncountable but has measure $\lambda(C) = 0$. This means that there are uncountable sets with measure zero.

In $\mathbb{R}^2$ and for two-dimensional Lebesgue measure λ^2 the situation is even easier: every line L in the plane has zero Lebesgue measure and L contains certainly uncountably many points. That $\lambda^2(L) = 0$ is seen from the fact that L differs from the ordinate $\{(x,y) \in \mathbb{R}^2 : x = 0\}$ only by a rigid motion T which leaves Lebesgue measure invariant (see Chapter 5) and $\lambda^2(\{x = 0\}) = 0$ as seen in Problem 6.4.

Problem 10.5 (i) Since $\{|u| > c\} \subset \{|u| \geqslant c\}$ and, therefore, $\mu(\{|u| > c\}) \leqslant \mu(\{|u| \geqslant c\})$, this follows immediately from Proposition 10.12. Alternatively, one could also mimic the proof of this Proposition or use part (iii) of the present problem with $\phi(t) = t$, $t \geqslant 0$.

(ii) This will follow from (iii) with $\phi(t) = t^p$, $t \geqslant 0$, since $\mu(\{|u| > c\}) \leqslant \mu(\{|u| \geqslant c\})$ as $\{|u| > c\} \subset \{|u| \geqslant c\}$.

(iii) We have, since ϕ is increasing,

$$\begin{aligned}
\mu(\{|u| \geqslant c\}) &= \mu(\{\phi(|u|) \geqslant \phi(c)\}) \\
&= \int \mathbb{1}_{\{x\,:\,\phi(|u(x)|)\geqslant\phi(c)\}}(x)\,\mu(dx) \\
&= \int \frac{\phi(|u(x)|)}{\phi(|u(x)|)}\mathbb{1}_{\{x\,:\,\phi(|u(x)|)\geqslant\phi(c)\}}(x)\,\mu(dx) \\
&\leqslant \int \frac{\phi(|u(x)|)}{\phi(c)}\mathbb{1}_{\{x\,:\,\phi(|u(x)|)\geqslant\phi(c)\}}(x)\,\mu(dx) \\
&\leqslant \int \frac{\phi(|u(x)|)}{\phi(c)}\,\mu(dx) \\
&= \frac{1}{\phi(c)}\int \phi(|u(x)|)\,\mu(dx)
\end{aligned}$$

(iv) Let us set $b = \alpha\int u\,d\mu$. Then we follow the argument of (iii):

$$\mu(\{u \geqslant b\}) = \int \mathbb{1}_{\{x\,:\,u(x)\geqslant b\}}(x)\,\mu(dx)$$

$$
\begin{aligned}
&= \int \frac{u(x)}{u(x)} \mathbb{1}_{\{x\,:\,u(x)\geqslant b\}}(x)\,\mu(dx) \\
&\leqslant \int \frac{u(x)}{b} \mathbb{1}_{\{x\,:\,u(x)\geqslant b\}}(x)\,\mu(dx) \\
&\leqslant \int \frac{u}{b}\,d\mu \\
&= \frac{1}{b}\int u\,d\mu
\end{aligned}
$$

and substituting $\alpha \int u\,d\mu$ for b shows the inequality.

(v) Using the fact that ψ is decreasing we get $\{|u| < c\} = \{\psi(|u|) > \psi(c)\}$—mind the change of the inequality sign—and going through the proof of part (iii) again we used there that ϕ increases only in the first step in a similar role as we used the decrease of ψ here! This means that the argument of (iii) is valid after this step and we get, altogether,

$$
\begin{aligned}
\mu(\{|u| < c\}) &= \mu(\{\psi(|u|) > \psi(c)\}) \\
&= \int \mathbb{1}_{\{x\,:\,\psi(|u(x)|)>\psi(c)\}}(x)\,\mu(dx) \\
&= \int \frac{\psi(|u(x)|)}{\psi(|u(x)|)} \mathbb{1}_{\{x\,:\,\psi(|u(x)|)>\phi(c)\}}(x)\,\mu(dx) \\
&\leqslant \int \frac{\psi(|u(x)|)}{\psi(c)} \mathbb{1}_{\{x\,:\,\psi(|u(x)|)>\psi(c)\}}(x)\,\mu(dx) \\
&\leqslant \int \frac{\psi(|u(x)|)}{\psi(c)}\,\mu(dx) \\
&= \frac{1}{\psi(c)}\int \psi(|u(x)|)\,\mu(dx)
\end{aligned}
$$

(vi) This follows immediately from (ii) by taking $\mu = P$, $c = \alpha\sqrt{VX}$, $u = X - EX$ and $p = 2$. Then

$$
\begin{aligned}
P(|X - EX| \geqslant \alpha EX) &\leqslant \frac{1}{(\alpha\sqrt{VX})^2}\int |X - EX|^2\,dP \\
&= \frac{1}{\alpha^2 VX}\,VX \;=\; \frac{1}{\alpha^2}.
\end{aligned}
$$

Problem 10.6 We mimic the proof of Corollary 10.13. Set $N = \{|u| = \infty\} = \{|u|^p = \infty\}$. Then $N = \bigcap_{k\in\mathbb{N}}\{|u|^p \geqslant k\}$ and using Markov's inequality (MI) and the 'continuity' of measures, Theorem 4.4, we find

$$
\begin{aligned}
\mu(N) = \mu\left(\bigcap_{k\in\mathbb{N}}\{|u|^p \geqslant k\}\right) &\overset{4.4}{=} \lim_{k\to\infty}\mu(\{|u|^p \geqslant k\}) \\
&\overset{MI}{\leqslant} \lim_{k\to\infty}\frac{1}{k}\underbrace{\int |u|^p\,d\mu}_{<\infty} \;=\; 0.
\end{aligned}
$$

For arctan this is not any longer true for several reasons:

- ... arctan is odd and changes sign, so there could be cancelations under the integral.

- ... even if we had no cancelations we have the problem that the points where $u(x) = \infty$ are now transformed to points where $\arctan(u(x)) = \frac{\pi}{2}$ and we do not know how the measure μ acts under this transformation. A simple example: Take μ to be a measure of total finite mass (that is: $\mu(X) < \infty$), e.g. a probability measure, and take the function $u(x)$ which is constantly $u \equiv +\infty$. Then $\arctan(u(x)) = \frac{\pi}{2}$ throughout, and we get
$$\int \arctan u(x)\,\mu(dx) = \int \frac{\pi}{2}\,d\mu = \frac{\pi}{2}\int d\mu = \frac{\pi}{2}\mu(X) < \infty,$$
but u is *nowhere* finite!

Problem 10.7 '$\Longrightarrow$': since the A_j are disjoint we get the identities

$$\mathbb{1}_{\dot\bigcup_j A_j} = \sum_{k=1}^{\infty} \mathbb{1}_{A_j} \qquad \text{and so} \qquad u\cdot\mathbb{1}_{\dot\bigcup_j A_j} = \sum_{k=1}^{\infty} u\cdot\mathbb{1}_{A_j},$$

hence $|u\mathbb{1}_{A_n}| = |u|\mathbb{1}_{A_n} \leqslant |u|\mathbb{1}_{\dot\bigcup_j A_j} = |u\mathbb{1}_{\dot\bigcup_j A_j}|$ showing the integrability of each $u\mathbb{1}_{A_n}$ by Theorem 10.3. By a Beppo Levi argument (Theorem 9.6) or, directly, by Corollary 9.9 we get

$$\begin{aligned}\sum_{j=1}^{\infty}\int_{A_j}|u|\,d\mu = \sum_{j=1}^{\infty}\int |u|\mathbb{1}_{A_j}\,d\mu &= \int \sum_{j=1}^{\infty}|u|\mathbb{1}_{A_j}\,d\mu \\ &= \int |u|\mathbb{1}_{\dot\bigcup_j A_j}\,d\mu \;<\; \infty.\end{aligned}$$

The converse direction '$\Longleftarrow$' follows again from Corollary 9.9, now just the other way round:

$$\begin{aligned}\int |u|\mathbb{1}_{\dot\bigcup_j A_j}\,d\mu = \int \sum_{j=1}^{\infty}|u|\mathbb{1}_{A_j}\,d\mu &= \sum_{j=1}^{\infty}\int |u|\mathbb{1}_{A_j}\,d\mu \\ &= \sum_{j=1}^{\infty}\int_{A_j}|u|\,d\mu \;<\; \infty\end{aligned}$$

showing that $u\mathbb{1}_{\dot\bigcup_j A_j}$ is integrable.

Problem 10.8 One possibility to solve the problem is to follow the hint. We go here a different (shorter) direction.

(i) Observe that $u_j - v \geqslant 0$ is a sequence of positive and integrable functions. Applying Fatou's lemma (in the usual form) yields (observing the rules for lim inf, lim sup from Appendix A, compare also Problem 9.8):

$$\begin{aligned}\int \liminf_j u_j\,d\mu - \int v\,d\mu &= \int \liminf_j (u_j - v)\,d\mu \\ &\leqslant \liminf_j \int (u_j - v)\,d\mu \\ &= \liminf_j \int u_j\,d\mu - \int v\,d\mu\end{aligned}$$

and the claim follows upon subtraction of the *finite (!)* number $\int v\,d\mu$.

(ii) Very similar to (i) by applying Fatou's lemma to the positive, integrable functions $w - u_j \geqslant 0$:

$$\begin{aligned}\int w\,d\mu - \int \limsup_j u_j\,d\mu &= \int \liminf_j (w - u_j)\,d\mu \\ &\leqslant \liminf_j \int (w - u_j)\,d\mu \\ &= \int w\,d\mu - \limsup_j \int u_j\,d\mu\end{aligned}$$

Now subtract the finite number $\int w\,d\mu$ on both sides.

(iii) We had the counterexample, in principle, already in Problem 9.8. Nevertheless...

Consider Lebesgue measure on $\mathbb{R}$. Put $f_j(x) = -\mathbb{1}_{[-2j,-j]}(x)$ and $g_j(x) = \mathbb{1}_{[j,2j]}(x)$. Then $\liminf f_j(x) = 0$ and $\limsup g_j(x) = 0$ for every x and neither admits an integrable minorant resp. majorant.

Problem 10.9 Note the misprint in the statement: the RHS should read $\sum_{j=0}^{\infty} P(|u| \geqslant j)$

We can safely assume that $u \geqslant 0$ (since integrability of u is equivalent to the integrability of $|u|$). Then

$$\begin{aligned}u(x) = \sum_{j=0}^{\infty} u(x)\mathbb{1}_{\{j\leqslant u<j+1\}}(x) &\geqslant \sum_{j=0}^{\infty} j\mathbb{1}_{\{j\leqslant u<j+1\}}(x) \\ &= \sum_{j=0}^{\infty} j\big(\mathbb{1}_{\{j\leqslant u\}}(x) - \mathbb{1}_{\{j+1\leqslant u\}}(x)\big).\end{aligned}$$

Since for fixed x, $u(x) < \infty$, we have $N\mathbb{1}_{\{N+1\leqslant u\}}(x) \xrightarrow{N\to\infty} 0$. Therefore, we can use Abel's summation trick and get

$$\begin{aligned}&\sum_{j=0}^{N} j\big(\mathbb{1}_{\{j\leqslant u\}}(x) - \mathbb{1}_{\{j+1\leqslant u\}}(x)\big) \\ &\quad = 0\cdot\big(\mathbb{1}_{\{0\leqslant u\}}(x) - \mathbb{1}_{\{1\leqslant u\}}(x)\big) + 1\cdot\big(\mathbb{1}_{\{1\leqslant u\}}(x) - \mathbb{1}_{\{2\leqslant u\}}(x)\big) \\ &\qquad + \cdots + N\cdot\big(\mathbb{1}_{\{N\leqslant u\}}(x) - \mathbb{1}_{\{N+1\leqslant u\}}(x)\big) \\ &\quad = \mathbb{1}_{\{1\leqslant u\}}(x) + \mathbb{1}_{\{2\leqslant u\}}(x) + \cdots + \mathbb{1}_{\{N\leqslant u\}}(x) - N\mathbb{1}_{\{N+1\leqslant u\}}(x)\end{aligned}$$

and this proves

$$\sum_{j=0}^{\infty} j\mathbb{1}_{\{j\leqslant u<j+1\}}(x) = \sum_{j=1}^{\infty} \mathbb{1}_{\{j\leqslant u\}}(x).$$

Therefore,

$$\begin{aligned}u = \sum_{j=0}^{\infty} u\,\mathbb{1}_{\{j\leqslant u<j+1\}} &\leqslant \sum_{j=0}^{\infty} (j+1)\mathbb{1}_{\{j\leqslant u<j+1\}} \\ &\leqslant \sum_{j=0}^{\infty} 2j\mathbb{1}_{\{j\leqslant u<j+1\}}\end{aligned}$$

$$= 2\sum_{j=1}^{\infty}\mathbb{1}_{\{j\leqslant u\}}(x) \;\leqslant\; 2u.$$

The claim follows from this, the fact that $\int \text{const.}\, dP = \text{const.}$ and Corollary 9.9:

$$\sum_{j=0}^{\infty} P(\{u \geqslant j\}) = \sum_{j=0}^{\infty}\int \mathbb{1}_{\{u\geqslant j\}}\, dP = \int \sum_{j=0}^{\infty}\mathbb{1}_{\{u\geqslant j\}}\, dP.$$

Problem 10.10 For $u = \mathbb{1}_B$ and $v = \mathbb{1}_C$ we have, because of independence,

$$\int uv\, dP = P(A\cap B) = P(A)P(B) = \int u\, dP \int v\, dP.$$

For positive, simple functions $u = \sum_j \alpha_j \mathbb{1}_{B_j}$ and $v = \sum_k \beta_k \mathbb{1}_{C_k}$ we find

$$\begin{aligned}\int uv\, dP &= \sum_{j,k}\alpha_j\beta_k \int \mathbb{1}_{A_j}\mathbb{1}_{B_k}\, dP\\ &= \sum_{j,k}\alpha_j\beta_k P(A_j\cap B_k)\\ &= \sum_{j,k}\alpha_j\beta_k P(A_j)P(B_k)\\ &= \Big(\sum_j \alpha_j P(A_j)\Big)\Big(\sum_k \beta_k P(B_k)\Big)\\ &= \int u\, dP \int v\, dP.\end{aligned}$$

For measurable $u \in \mathcal{M}^+(\mathcal{B})$ and $v\in\mathcal{M}^+(\mathcal{C})$ we use approximating simple functions $u_k \in \mathcal{E}^+(\mathcal{B})$, $u_k \uparrow u$, and $v_k \in \mathcal{E}^+(\mathcal{C})$, $v_k\uparrow v$. Then, by Beppo Levi,

$$\begin{aligned}\int uv\, dP = \lim_k \int u_k v_k\, dP &= \lim_k \int u_k\, dP \lim_j \int v_j\, dP\\ &= \int u\, dP \int v\, dP.\end{aligned}$$

Integrable independent functions: If $u\in\mathcal{L}^1(\mathcal{B})$ and $v\in\mathcal{L}^1(\mathcal{C})$, the above calculation when applied to $|u|, |v|$ shows that $u\cdot v$ is integrable since

$$\int |uv|\, dP \leqslant \int |u|\, dP \int |v|\, dP < \infty.$$

Considering positive and negative parts finally also gives

$$\int uv\, dP = \int u\, dP \int v\, dP.$$

Counterexample: Just take $u = v$ which are integrable but not square integrable, e.g. $u(x) = v(x) = x^{-1/2}$. Then $\int_{(0,1)} x^{-1/2}\, dx < \infty$ but $\int_{(0,1)} x^{-1}\, dx = \infty$, compare also Problem 10.2.

Problem 10.11 (i) Assume that f^* is $\mathcal{A}^*$-measurable. The problem at hand is to construct $\mathcal{A}$-measurable upper and lower functions g and f. For positive simple functions this is clear: if $f^*(x) = \sum_{j=0}^N \phi_j \mathbb{1}_{B_j^*}(x)$ with $\phi_j \geqslant 0$ and $B_j^* \in \mathcal{A}^*$, then we can use Problem 4.13(v) to find $B_j, C_j \in \mathcal{A}$ with $\mu(C_j \setminus B_j) = 0$

$$B_j \subset B_j^* \subset C_j \implies \phi_j \mathbb{1}_{B_j} \leqslant \phi_j \mathbb{1}_{B_j^*} \leqslant \phi_j \mathbb{1}_{C_j}$$

and summing over $j = 0, 1, \dots, N$ shows that $f \leqslant f^* \leqslant g$ where f, g are the appropriate lower and upper sums which are clearly $\mathcal{A}$ measurable and satisfy

$$\begin{aligned}\mu(\{f \neq g\}) &\leqslant \mu(C_0 \setminus B_0 \cup \dots \cup C_N \setminus B_N)\\ &\leqslant \mu(C_0 \setminus B_0) + \dots + \mu(C_N \setminus B_N)\\ &= 0 + \dots + 0 = 0.\end{aligned}$$

Moreover, since by Problem 4.13 $\mu(B_j) = \mu(C_j) = \bar{\mu}(B_j^*)$, we have

$$\sum_j \phi_j \mu(B_j) = \sum_j \phi_j \bar{\mu}(B_j^*) = \sum_j \phi_j \mu(C_j)$$

which is the same as

$$\int f \, d\mu = \int f^* \, d\bar{\mu} = \int g \, d\mu.$$

(ii), (iii) Assume that u^* is $\mathcal{A}^*$-measurable; without loss of generality (otherwise consider positive and negative parts) we can assume that $u^* \geqslant 0$. Because of Theorem 8.8 we know that $f_k^* \uparrow u^*$ for $f_k^* \in \mathcal{E}^+(\mathcal{A}^*)$. Now choose the corresponding $\mathcal{A}$-measurable lower and upper functions f_k, g_k constructed in part (i). By considering, if necessary, $\max\{f_1, \dots, f_k\}$ we can assume that the f_k are increasing.

Set $u := \sup_k f_k$ and $v := \liminf_k g_k$. Then $u, v \in \mathcal{M}(\mathcal{A})$, $u \leqslant u^* \leqslant v$, and by Fatou's lemma

$$\begin{aligned}\int v \, d\mu = \int \liminf_k g_k \, d\mu &\leqslant \liminf_k \int g_k \, d\mu\\ &= \liminf_k \int f_k^* \, d\bar{\mu}\\ &= \int u^* \, d\bar{\mu}\\ &\leqslant \int v \, d\mu.\end{aligned}$$

Since $f_k \uparrow u$ we get by Beppo Levi and Fatou

$$\begin{aligned}\int u \, d\mu = \sup_k \int f_k \, d\mu = \liminf_k \int f_k \, d\mu &= \liminf_k \int g_k \, d\mu\\ &\geqslant \int \liminf_k g_k \, d\mu\\ &= \int v \, d\mu\\ &\geqslant \int u \, d\mu\end{aligned}$$

This proves that $\int u\,d\mu = \int v\,d\mu = \int u^*\,d\mu$. This answers part (iii) by considering positive and negative parts.

It remains to show that $\{u \neq v\}$ is a μ-null set. (This does not follow from the above integral equality, cf. Problem 10.16!) Clearly, $\{u \neq v\} = \{u < v\}$, i.e. if $x \in \{u < v\}$ is fixed, we deduce that, for sufficiently large values of k,

$$f_k(x) < g_k(x), \quad k \text{ large}$$

since $u = \sup f_k$ and $v = \liminf_k g_k$. Thus,

$$\{u \neq v\} \subset \bigcup_k \{f_k \neq g_k\}$$

but the RHS is a countable union of μ-null sets, hence a null set itself.

Conversely, assume first that $u \leqslant u^* \leqslant v$ for two $\mathcal{A}$-measurable functions u, v with $u = v$ a.e. We have to show that $\{u^* > \alpha\} \in \mathcal{A}^*$. Using that $u \leqslant u^* \leqslant v$ we find that

$$\{u > \alpha\} \subset \{u^* > \alpha\} \subset \{v > \alpha\}$$

but $\{v > \alpha\}, \{u > \alpha\} \in \mathcal{A}$ and $\{u > \alpha\} \setminus \{v > \alpha\} \subset \{u \neq v\}$ is a μ-null set. Because of Problem 4.13 we conclude that $\{u^* > \alpha\} \in \mathcal{A}^*$.

Problem 10.12 Note the misprint in the statement: for the estimate $\mu_*(E) + \mu_*(F) \leqslant \mu_*(E \cup F)$ the sets E, F should be disjoint!

Throughout the solution the letters A, B are reserved for sets from $\mathcal{A}$.

(i) a) Let $A \subset E \subset B$. Then $\mu(A) \leqslant \mu(B)$ and going to the $\sup_{A \subset E}$ and $\inf_{E \subset B}$ proves $\mu_*(E) \leqslant \mu^*(E)$.

b) By the definition of μ_* and μ^* we find some $A \subset E$ such that

$$|\mu_*(E) - \mu(A)| \leqslant \epsilon.$$

Since $A^c \supset E^c$ we can enlarge A, if needed, and achieve

$$|\mu^*(E^c) - \mu(A^c)| \leqslant \epsilon.$$

Thus,

$$\begin{aligned} &|\mu(X) - \mu_*(E) - \mu^*(E^c)| \\ &\qquad \leqslant |\mu_*(E) - \mu(A)| + |\mu^*(E^c) - \mu(A^c)| \\ &\qquad \leqslant 2\epsilon, \end{aligned}$$

and the claim follows as $\epsilon \to 0$.

c) Let $A \supset E$ and $B \supset F$ be arbitrary majorizing $\mathcal{A}$-sets. Then $A \cup B \supset E \cup F$ and

$$\mu^*(E \cup F) \leqslant \mu(A \cup B) \leqslant \mu(A) + \mu(B).$$

Now we pass on the right-hand side, separately, to the $\inf_{A\supset E}$ and $\inf_{B\supset F}$, and obtain

$$\mu^*(E \cup F) \leqslant \mu^*(E) + \mu^*(F).$$

d) Let $A \subset E$ and $B \subset F$ be arbitrary minorizing $\mathcal{A}$-sets. Then $A \uplus B \subset E \uplus F$ and

$$\mu_*(E \uplus F) \geqslant \mu(A \uplus B) = \mu(A) + \mu(B).$$

Now we pass on the right-hand side, separately, to the $\sup_{A\supset E}$ and $\sup_{B\supset F}$, where we stipulate that $A \cap B = \varnothing$, and obtain

$$\mu_*(E \uplus F) \geqslant \mu_*(E) + \mu_*(F).$$

(ii) By the definition of the infimum/supremum we find sets $A_n \subset E \subset A^n$ such that

$$|\mu_*(A) - \mu(A_n)| + |\mu^*(A) - \mu(A^n)| \leqslant \frac{1}{n}.$$

Without loss of generality we can assume that the A_n increase and that the A^n decrease. Now $A_* := \bigcup_n A_n$, $A^* := \bigcap_n A^n$ are $\mathcal{A}$-sets with $A_* \subset A \subset A^*$. Now, $\mu(A^n) \downarrow \mu(A^*)$ as well as $\mu(A^n) \to \mu^*(E)$ which proves $\mu(A^*) = \mu^*(E)$. Analogously, $\mu(A_n) \uparrow \mu(A_*)$ as well as $\mu(A_n) \to \mu_*(E)$ which proves $\mu(A_*) = \mu_*(E)$.

(iii) In view of Problem 4.13 and (i), (ii), it is clear that

$$\big\{E \subset X \,:\, \mu_*(E) = \mu^*(E)\big\} =$$
$$\big\{E \subset X \,:\, \exists\, A, B \in \mathcal{A},\ A \subset E \subset B,\ \mu(B \setminus A) = 0\big\}$$

but the latter is the completed σ-algebra $\mathcal{A}^*$. That $\mu^*\big|_{\mathcal{A}^*} = \mu_*\big|_{\mathcal{A}^*} = \bar{\mu}$ is now trivial since μ_* and μ^* coincide on $\mathcal{A}^*$.

Problem 10.13 Let $A \in \mathcal{A}$ and assume that there are non-measurable sets, i.e. $\mathcal{P}(X) \supsetneq \mathcal{A}$. Take some $N \notin \mathcal{A}$ which is a μ-null set. Assume also that $N \cap A = \varnothing$. Then $u = \mathbb{1}_A$ and $w := \mathbb{1}_A + 2 \cdot \mathbb{1}_N$ are a.e. identical, but w is not measurable.

This means that w is only measurable if, e.g. all (subsets of) null sets are measurable, that is if $(X, \mathcal{A}, \mu)$ is complete.

Problem 10.14 The function $\mathbb{1}_{\mathbb{Q}}$ is nowhere continuous but $u = 0$ Lebesgue almost everywhere. That is

$$\{x \,:\, \mathbb{1}_{\mathbb{Q}}(x) \ \text{ is discontinuous}\} = \mathbb{R}$$

while

$$\{x \,:\, \mathbb{1}_{\mathbb{Q}} \neq 0\} = \mathbb{Q} \ \text{ is a Lebesgue null set,}$$

that is $\mathbb{1}_{\mathbb{Q}}$ coincides a.e. with a continuous function but is itself at no point continuous!

The same analysis for $\mathbb{1}_{[0,\infty)}$ yields that

$$\{x \,:\, \mathbb{1}_{[0,\infty)}(x) \ \text{ is discontinuous}\} = \{0\}$$

which is a Lebesgue null set, but $\mathbb{1}_{[0,\infty)}$ cannot coincide a.e. with a continuous function! This, namely, would be of the form $w = 0$ on $(-\infty, -\delta)$ and $w = 1$ on (ϵ, ∞) while it 'interpolates' somehow between 0 and 1 if $-\delta < x < \epsilon$. But this entails that

$$\{x : w(x) \neq \mathbb{1}_{[0,\infty)}(x)\}$$

cannot be a Lebesgue null set!

Problem 10.15 Let $(A_j)_{j\in\mathbb{N}} \subset \mathcal{A}$ be an exhausting sequence $A_j \uparrow X$ such that $\mu(A_j) < \infty$. Set

$$f(x) := \sum_{j=1}^{\infty} \frac{1}{2^j(\mu(A_j)+1)} \mathbb{1}_{A_j}(x).$$

Then f is measurable, $f(x) > 0$ everywhere, and using Beppo Levi's theorem

$$\begin{aligned}\int f\,d\mu &= \int \left(\sum_{j=1}^{\infty} \frac{1}{2^j(\mu(A_j)+1)} \mathbb{1}_{A_j}\right) d\mu \\ &= \sum_{j=1}^{\infty} \frac{1}{2^j(\mu(A_j)+1)} \int \mathbb{1}_{A_j}\,d\mu \\ &= \sum_{j=1}^{\infty} \frac{\mu(A_j)}{2^j(\mu(A_j)+1)} \\ &\leqslant \sum_{j=1}^{\infty} 2^{-j} = 1.\end{aligned}$$

Thus, set $P(A) := \int_A f\,d\mu$. We know from Problem 9.5 that P is indeed a measure.

If $N \in \mathcal{N}_\mu$, then, by Theorem 10.9,

$$P(N) = \int_N f\,d\mu \overset{10.9}{=} 0$$

so that $\mathcal{N}_\mu \subset \mathcal{N}_P$.

Conversely, if $M \in \mathcal{M}_P$, we see that

$$\int_M f\,d\mu = 0$$

but since $f > 0$ everywhere, it follows from Theorem 10.9 that $\mathbb{1}_M \cdot f = 0$ μ-a.e., i.e. $\mu(M) = 0$. Thus, $\mathcal{N}_P \subset \mathcal{N}_\mu$.

Remark. We will see later (cf. Chapter 19, Radon-Nikodým theorem) that $\mathcal{N}_\mu = \mathcal{N}_P$ if and only if $P = f \cdot \mu$ (i.e., if P has a density w.r.t. μ) such that $f > 0$.

Problem 10.16 Well, the hint given in the text should be good enough.

11 Convergence theorems and their applications. Solutions to Problems 11.1–11.21

Problem 11.1 We start with the simple remark that

$$\begin{aligned} |a-b|^p &\leqslant (|a|+|b|)^p \\ &\leqslant (\max\{|a|,|b|\}+\max\{|a|,|b|\})^p \\ &= 2^p \max\{|a|,|b|\}^p \\ &= 2^p \max\{|a|^p,|b|^p\} \\ &\leqslant 2^p(|a|^p+|b|^p). \end{aligned}$$

Because of this we find that $|u_j - u|^p \leqslant 2^p g^p$ and the right-hand side is an integrable dominating function.

Proof alternative 1: Apply Theorem 11.2 on dominated convergence to the sequence $\phi_j := |u_j - u|^p$ of integrable functions. Note that $\phi_j(x) \to 0$ and that $0 \leqslant \phi_j \leqslant \Phi$ where $\Phi = 2^p g^p$ is integrable and independent of j. Thus,

$$\begin{aligned} \lim_{j\to\infty} \int |u_j - u|^p \, d\mu = \lim_{j\to\infty} \int \phi_j \, d\mu &= \int \lim_{j\to\infty} \phi_j \, d\mu \\ &= \int 0 \, d\mu \; = \; 0. \end{aligned}$$

Proof alternative 2: Mimic the proof of Theorem 11.2 on dominated convergence. To do so we remark that the sequence of functions

$$0 \leqslant \psi_j := 2^p g^p - |u_j - u|^p \xrightarrow{j\to\infty} 2^p g^p$$

Since the limit $\lim_j \psi_j$ exists, it coincides with $\liminf_j \psi_j$, and so we can use Fatou's Lemma to get

$$\begin{aligned} \int 2^p g^p \, d\mu &= \int \liminf_{j\to\infty} \psi_j \, d\mu \\ &\leqslant \liminf_{j\to\infty} \int \psi_j \, d\mu \\ &= \liminf_{j\to\infty} \int \left(2^p g^p - |u_j - u|^p\right) d\mu \end{aligned}$$

$$= \int 2^p g^p \, d\mu + \liminf_{j\to\infty} \left(- \int |u_j - u|^p \, d\mu \right)$$
$$= \int 2^p g^p \, d\mu - \limsup_{j\to\infty} \int |u_j - u|^p \, d\mu$$

where we used that $\liminf_j(-\alpha_j) = -\limsup_j \alpha_j$. This shows that $\limsup_j \int |u_j - u|^p \, d\mu = 0$, hence

$$0 \leqslant \liminf_{j\to\infty} \int |u_j - u|^p \, d\mu \leqslant \limsup_{j\to\infty} \int |u_j - u|^p \, d\mu \leqslant 0$$

showing that lower and upper limit coincide and equal to 0, hence $\lim_j \int |u_j - u|^p \, d\mu = 0$.

Problem 11.2 Assume that, as in the statement of Theorem 11.2, $u_j \to u$ and that $|u_j| \leqslant f \in \mathcal{L}^1(\mu)$. In particular,

$$-f \leqslant u_j \quad \text{and} \quad u_j \leqslant f$$

($j \in \mathbb{N}$) is an integrable minorant resp. majorant. Thus, using Problem 10.8 at $*$ below,

$$\begin{aligned}\int u \, d\mu &= \int \liminf_{j\to\infty} u_j \, d\mu \\ &\overset{*}{\leqslant} \liminf_{j\to\infty} \int u_j \, d\mu \\ &\leqslant \limsup_{j\to\infty} \int u_j \, d\mu \\ &\overset{*}{\leqslant} \int \limsup_{j\to\infty} u_j \, d\mu = \int u \, d\mu.\end{aligned}$$

This proves $\int u \, d\mu = \lim_j \int u_j \, d\mu$.

Addition: since $0 \leqslant |u - u_j| \leqslant |\lim_j u_j| + |u_j| \leqslant 2f \in \mathcal{L}^1(\mu)$, the sequence $|u - u_j|$ has an integrable majorant and using Problem 10.8 we get

$$0 \leqslant \limsup_{j\to\infty} \int |u_j - u| \, d\mu \leqslant \int \limsup_{j\to\infty} |u_j - u| \, d\mu = \int 0 \, d\mu = 0$$

and also (i) of Theorem 11.2 follows...

Problem 11.3 By assumption we have

$$0 \leqslant f_k - g_k \xrightarrow{k\to\infty} f - g,$$
$$0 \leqslant G_k - f_k \xrightarrow{k\to\infty} G - f.$$

Using Fatou's Lemma we find

$$\begin{aligned}\int (f - g) \, d\mu &= \int \lim_k (f_k - g_k) \, d\mu \\ &= \int \liminf_k (f_k - g_k) \, d\mu \\ &\leqslant \liminf_k \int (f_k - g_k) \, d\mu \\ &= \liminf_k \int f_k \, d\mu - \int g \, d\mu,\end{aligned}$$

and

$$\begin{aligned}\int (G-f)\,d\mu &= \int \lim_k (G_k - f_k)\,d\mu \\ &= \int \liminf_k (G_k - f_k)\,d\mu \\ &\leqslant \liminf_k \int (G_k - f_k)\,d\mu \\ &= \int G\,d\mu - \limsup_k \int f_k\,d\mu.\end{aligned}$$

Adding resp. subtracting $\int g\,d\mu$ resp. $\int G\,d\mu$ therefore yields

$$\limsup_k \int f_k\,d\mu \leqslant \int f\,d\mu \leqslant \liminf_k \int f_k\,d\mu$$

and the claim follows.

Problem 11.4 Using Beppo Levi's theorem in the form of Corollary 9.9 we find

$$\int \sum_{j=1}^{\infty} |u_j|\,d\mu = \sum_{j=1}^{\infty} \int |u_j|\,d\mu < \infty, \tag{*}$$

which means that the positive function $\sum_{j=1}^{\infty} |u_j|$ is finite almost everywhere, i.e. the series $\sum_{j=1}^{\infty} u_j$ converges (absolutely) almost everywhere.

Moreover,

$$\int \sum_{j=1}^{N} u_j\,d\mu = \sum_{j=1}^{N} \int u_j\,d\mu \tag{**}$$

and, using the triangle inequality both quantities

$$\left| \int \sum_{j=n}^{N} u_j\,d\mu \right| \quad\text{and}\quad \left| \sum_{j=n}^{N} \int u_j\,d\mu \right|$$

can be estimated by

$$\int \sum_{j=n}^{N} |u_j|\,d\mu \xrightarrow{n,N\to\infty} 0$$

because of (*). This shows that both sides in (**) are Cauchy sequences, i.e. they are convergent.

Problem 11.5 Since $\mathcal{L}^1(\mu) \ni u_j \downarrow 0$ we find by monotone convergence, Theorem 11.1, that $\int u_j\,d\mu \downarrow 0$. Therefore,

$$\sigma = \sum_{j=1}^{\infty} (-1)^j u_j \quad\text{and}\quad S = \sum_{j=1}^{\infty} (-1)^j \int u_j\,d\mu \quad \text{converge}$$

(conditionally, in general). Moreover, for every $N \in \mathbb{N}$,

$$\int \sum_{j=1}^{N} (-1)^j u_j\,d\mu = \sum_{j=1}^{N} \int (-1)^j u_j\,d\mu \xrightarrow{N\to\infty} S.$$

All that remains is to show that the right-hand side converges to $\int \sigma\,d\mu$. Observe that for $S_N := \sum_{j=1}^N (-1)^j u_j$ we have

$$S_{2N} \leqslant S_{2N+2} \leqslant \ldots \leqslant S$$

and we find, as $S_j \in \mathcal{L}^1(\mu)$, by monotone convergence that

$$\lim_{N\to\infty} \int S_{2N}\,d\mu = \int \sigma\,d\mu.$$

Problem 11.6 Consider $u_j(x) := j \cdot \mathbb{1}_{(0,1/j)}(x)$, $j \in \mathbb{N}$. It is clear that u_j is measurable and Lebesgue integrable with integral

$$\int u_j\,d\lambda = j\,\frac{1}{j} = 1 \qquad \forall\, j \in \mathbb{N}.$$

Thus, $\lim_j \int u_j\,d\lambda = 1$. On the other hand, the pointwise limit is

$$u(x) := \lim_j u_j(x) \equiv 0$$

so that $0 = \int u\,d\lambda = \int \lim_j u_j\,d\lambda \neq 1$.

The example does not contradict dominated convergence as there is no uniform dominating integrable function.

Alternative: a similar situation can be found for $v_k(x) := \frac{1}{k}\,\mathbb{1}_{[0,k]}(x)$ and the pointwise limit $v \equiv 0$. Note that in this case the limit is even uniform and still $\lim_k \int v_k\,d\lambda = 1 \neq 0 = \int v\,d\lambda$. Again there is no contradiction to dominated convergence as there does not exist a uniform dominating integrable function.

Problem 11.7 Let μ be an arbitrary Borel measure on the line $\mathbb{R}$ and define the integral function for some $u \in \mathcal{L}^1(\mu)$ through

$$I(x) := I_\mu^u(x) := \int_{(0,x)} u(t)\,\mu(dt) = \int \mathbb{1}_{(0,x)}(t)u(t)\,\mu(dt).$$

For any sequence $0 < l_j \to x$, $l_j < x$ from the left and $r_k \to x$, $r_k > x$ from the right we find

$$\mathbb{1}_{(0,l_j)}(t) \xrightarrow{j\to\infty} \mathbb{1}_{(0,x)}(t) \quad\text{and}\quad \mathbb{1}_{(0,r_k)}(t) \xrightarrow{k\to\infty} \mathbb{1}_{(0,x]}(t).$$

Since $|\mathbb{1}_{(0,x)}u| \leqslant |u| \in \mathcal{L}^1$ is a uniform dominating function, Lebesgue's dominated convergence theorem yields

$$\begin{aligned}
I(x+) - I(x-) &= \lim_k I(r_k) - \lim_j I(l_j) \\
&= \int \mathbb{1}_{(0,x]}(t)u(t)\,\mu(dt) - \int \mathbb{1}_{(0,x)}(t)u(t)\,\mu(dt) \\
&= \int \big(\mathbb{1}_{(0,x]}(t) - \mathbb{1}_{(0,x)}(t)\big)u(t)\,\mu(dt) \\
&= \int \mathbb{1}_{\{x\}}(t)u(t)\,\mu(dt) \\
&= u(x)\,\mu(\{x\}).
\end{aligned}$$

Thus $I(x)$ is continuous at x if, and only if, x is not an atom of μ.

Remark: the proof shows, by the way, that $I_\mu^u(x)$ is *always* left-continous at every x, no matter what μ or u look like.

Problem 11.8 (i) We have

$$\begin{aligned}
&\int \frac{1}{x}\,\mathbb{1}_{[1,\infty)}(x)\,dx \\
&\quad = \lim_{n\to\infty} \int \frac{1}{x}\,\mathbb{1}_{[1,n)}(x)\,dx && \text{by Beppo Levi's thm.} \\
&\quad = \lim_{n\to\infty} \int_{[1,n)} \frac{1}{x}\,dx && \text{usual shorthand} \\
&\quad = \lim_{n\to\infty} (R)\int_1^n \frac{1}{x}\,dx && \text{Riemann-}\int_1^n \text{ exists} \\
&\quad = \lim_{n\to\infty} \left[\log x\right]_1^n \\
&\quad = \lim_{n\to\infty} [\log(n) - \log(1)] = \infty
\end{aligned}$$

which means that $\frac{1}{x}$ is not Lebesgue-integrable over $[1,\infty)$.

(ii) We have

$$\begin{aligned}
&\int \frac{1}{x^2}\,\mathbb{1}_{[1,\infty)}(x)\,dx \\
&\quad = \lim_{n\to\infty} \int \frac{1}{x^2}\,\mathbb{1}_{[1,n)}(x)\,dx && \text{by Beppo Levi's thm.} \\
&\quad = \lim_{n\to\infty} \int_{[1,n)} \frac{1}{x^2}\,dx && \text{usual shorthand} \\
&\quad = \lim_{n\to\infty} (R)\int_1^n \frac{1}{x^2}\,dx && \text{Riemann-}\int_1^n \text{ exists} \\
&\quad = \lim_{n\to\infty} \left[-\tfrac{1}{x}\right]_1^n \\
&\quad = \lim_{n\to\infty} [1 - \tfrac{1}{n}] = 1 < \infty
\end{aligned}$$

which means that $\frac{1}{x^2}$ is Lebesgue-integrable over $[1,\infty)$.

(iii) We have

$$\begin{aligned}
&\int \frac{1}{\sqrt{x}}\,\mathbb{1}_{(0,1]}(x)\,dx \\
&\quad = \lim_{n\to\infty} \int \frac{1}{\sqrt{x}}\,\mathbb{1}_{(1/n,1]}(x)\,dx && \text{by Beppo Levi's thm.} \\
&\quad = \lim_{n\to\infty} \int_{(1/n,1]} \frac{1}{\sqrt{x}}\,dx && \text{usual shorthand} \\
&\quad = \lim_{n\to\infty} (R)\int_{1/n}^1 \frac{1}{\sqrt{x}}\,dx && \text{Riemann-}\int_{1/n}^1 \text{ exists} \\
&\quad = \lim_{n\to\infty} \left[2\sqrt{x}\right]_{1/n}^1 \\
&\quad = \lim_{n\to\infty} \left[2 - 2\sqrt{\tfrac{1}{n}}\right] \\
&\quad = 2 < \infty
\end{aligned}$$

which means that $\frac{1}{\sqrt{x}}$ is Lebesgue-integrable over $(0,1]$.

(iv) We have

$$\int \frac{1}{x}\,\mathbb{1}_{(0,1]}(x)\,dx$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \int \frac{1}{x} \mathbb{1}_{(1/n,1]}(x)\,dx && \text{by Beppo Levi's thm.}\\
&= \lim_{n\to\infty} \int_{(1/n,1]} \frac{1}{x}\,dx && \text{usual shorthand}\\
&= \lim_{n\to\infty} (R) \int_{1/n}^{1} \frac{1}{x}\,dx && \text{Riemann-}\int_{1/n}^{1} \text{ exists}\\
&= \lim_{n\to\infty} [\log x]_{1/n}^{1}\\
&= \lim_{n\to\infty} \left[\log(1) - \log \tfrac{1}{n}\right]\\
&= \infty
\end{aligned}
$$

which means that $\frac{1}{x}$ is not Lebesgue-integrable over $(0,1]$.

Problem 11.9 We construct a dominating integrable function.

If $x \leqslant 1$, we have clearly $\exp(-x^\alpha) \leqslant 1$, and $\int_{(0,1]} \mathbb{1}\,dx = 1 < \infty$ is integrable.

If $x \geqslant 1$, we have $\exp(-x^\alpha) \leqslant Mx^{-2}$ for some suitable constant $M = M_\alpha < \infty$. This function is integrable in $[1,\infty)$, see e.g. Problem 11.8. The estimate is easily seen from the fact that $x \mapsto x^2 \exp(-x^\alpha)$ is continuous in $[1,\infty)$ with $\lim_{x\to\infty} x^2 \exp(-x^\alpha) = 0$.

This shows that $\exp(-x^\alpha) \leqslant \mathbb{1}_{(0,1)} + Mx^{-2}\,\mathbb{1}_{[1,\infty)}$ with the right-hand side being integrable.

Problem 11.10 Take $\alpha \in (a,b)$ where $0 < a < b < \infty$ are fixed (but arbitrary). We show that the function is continuous for these α. This shows the general case since continuity is a local property and we can 'catch' any given α_0 by some choice of a and b's.

We use the Continuity lemma (Theorem 11.4) and have to find uniform (for $\alpha \in (a,b)$) dominating bounds on the integrand function $f(\alpha,x) := \left(\frac{\sin x}{x}\right)^3 e^{-\alpha x}$. First of all, we remark that $\left|\frac{\sin x}{x}\right| \leqslant M$ which follows from the fact that $\frac{\sin x}{x}$ is a continuous function such that $\lim_{x\to\infty} \frac{\sin x}{x} = 0$ and $\lim_{x\downarrow 0} \frac{\sin x}{x} = 1$. (Actually, we could choose $M = 1$...). Moreover, $\exp(-\alpha x) \leqslant 1$ for $x \in (0,1)$ and $\exp(-\alpha x) \leqslant C_{a,b}x^{-2}$ for $x \geqslant 1$—use for this the continuity of $x^2\exp(-\alpha x)$ and the fact that $\lim_{x\to\infty} x^2 \exp(-\alpha x) = 0$. This shows that

$$
|f(\alpha,x)| \leqslant M\left(\mathbb{1}_{(0,1)}(x) + C_{a,b}x^{-2}\,\mathbb{1}_{[1,\infty)}(x)\right)
$$

and the right-hand side is an integrable dominating function which does not depend on α—as long as $\alpha \in (a,b)$. But since $\alpha \mapsto f(\alpha,x)$ is obviously continuous, the Continuity lemma applies and proves that $\int_{(0,\infty)} f(\alpha,x)\,dx$ is continuous.

Problem 11.11 Fix some number $N > 0$ and take $x \in (-N,N)$. We show that $G(x)$ is continuous on this set. Since N was arbitrary, we find that G is continuous for every $x \in \mathbb{R}$.

Set $g(t,x) := \frac{\sin(tx)}{t(1+t^2)} = x\,\frac{\sin(tx)}{(tx)}\,\frac{1}{1+t^2}$. Then, using that $\left|\frac{\sin u}{u}\right| \leqslant M$, we have

$$
|g(t,x)| \leqslant x \cdot M \cdot \frac{1}{1+t^2} \leqslant M \cdot N \cdot \left(\mathbb{1}_{(0,1)}(t) + \frac{1}{t^2}\,\mathbb{1}_{[1,\infty)}(t)\right)
$$

and the right-hand side is a uniformly dominating function, i.e. $G(x)$ makes sense and we find $G(0) = \int_{t\neq 0} g(t,0)\,dt = 0$. To see differentiability, we use the Differentiability lemma

(Theorem 11.5) and need to prove that $|\partial_x g(t,x)|$ exists (this is clear) and is uniformly dominated for $x \in (-N,N)$. We have

$$\begin{aligned}|\partial_x g(t,x)| = \left|\partial_x \frac{\sin(tx)}{t(1+t^2)}\right| &= \left|\frac{\cos(tx)}{(1+t^2)}\right| \\ &\leqslant \frac{1}{1+t^2} \\ &\leqslant \left(\mathbb{1}_{(0,1)}(t) + \frac{1}{t^2}\mathbb{1}_{[1,\infty)}(t)\right)\end{aligned}$$

and this allows us to apply the Differentiability lemma, so

$$\begin{aligned}G'(x) = \partial_x \int_{t\neq 0} g(t,x)\,dt &= \int_{t\neq 0} \partial_x g(t,x)\,dt \\ &= \int_{t\neq 0} \frac{\cos(tx)}{1+t^2}\,dt \\ &= \int_{\mathbb{R}} \frac{\cos(tx)}{1+t^2}\,dt\end{aligned}$$

(use in the last equality that $\{0\}$ is a Lebesgue null set). Thus, by a Beppo-Levi argument (and using that Riemann=Lebesgue whenever the Riemann integral over a compact interval exists...)

$$\begin{aligned}G'(0) = \int_{\mathbb{R}} \frac{1}{1+t^2}\,dt &= \lim_{n\to\infty}(R)\int_{-n}^{n} \frac{1}{1+t^2}\,dt \\ &= \lim_{n\to\infty}\left[\tan^{-1}(t)\right]_{-n}^{n} \\ &= \pi.\end{aligned}$$

Now observe that

$$\partial_x \sin(tx) = t\cos(tx) = \frac{t}{x}\,x\cos(tx) = \frac{t}{x}\,\partial_t \sin(tx).$$

Since the integral defining $G'(x)$ exists we can use a Beppo-Levi argument, Riemann = Lebesgue (whenever the Riemann integral over an interval exists) and integration by parts (for the Riemann integral) to find

$$\begin{aligned}xG'(x) &= \int_{\mathbb{R}} \frac{x\cos(tx)}{1+t^2}\,dt \\ &= \lim_{n\to\infty}(R)\int_{-n}^{n} \frac{x\partial_x \sin(tx)}{t(1+t^2)}\,dt \\ &= \lim_{n\to\infty}(R)\int_{-n}^{n} \frac{t\partial_t \sin(tx)}{t(1+t^2)}\,dt \\ &= \lim_{n\to\infty}(R)\int_{-n}^{n} \frac{\partial_t \sin(tx)}{1+t^2}\,dt \\ &= \lim_{n\to\infty}(R)\int_{-n}^{n} \partial_t \sin(tx)\cdot\frac{1}{1+t^2}\,dt \\ &= \lim_{n\to\infty}\left[\frac{\sin(tx)}{1+t^2}\right]_{t=-n}^{n} - \lim_{n\to\infty}(R)\int_{-n}^{n} \sin(tx)\cdot\partial_t\frac{1}{1+t^2}\,dt \\ &= \lim_{n\to\infty}(R)\int_{-n}^{n} \sin(tx)\cdot\frac{2t}{(1+t^2)^2}\,dt \\ &= \int_{\mathbb{R}} \frac{2t\sin(tx)}{(1+t^2)^2}\,dt.\end{aligned}$$

Problem 11.12 (i) Note that for $0 \leqslant a, b \leqslant 1$

$$1 - (1-a)^b = \int_{1-a}^1 bt^{b-1}\,dt \geqslant \int_{1-a}^1 b\,dt = ba$$

so that we get for $0 \leqslant x \leqslant k$ and $a := x/k$, $b := k/(k+1)$

$$\left(1 - \frac{x}{k}\right)^{\frac{k}{k+1}} \leqslant 1 - \frac{x}{k+1}, \qquad 0 \leqslant x \leqslant k$$

or,

$$\left(1 - \frac{x}{k}\right)^k \mathbb{1}_{[0,k]}(x) \leqslant \left(1 - \frac{x}{k+1}\right)^{k+1} \mathbb{1}_{[0,k+1]}(x).$$

Therefore we can appeal to Beppo Levi's theorem to get

$$\begin{aligned}\lim_{k\to\infty} \int_{(1,k)} \left(1 - \frac{x}{k}\right)^k \ln x\,\lambda^1(dx) &= \sup_{k\in\mathbb{N}} \int \mathbb{1}_{(1,k)}(x)\left(1 - \frac{x}{k}\right)^k \ln x\,\lambda^1(dx)\\ &= \int \sup_{k\in\mathbb{N}} \left[\mathbb{1}_{(1,k)}(x)\left(1 - \frac{x}{k}\right)^k\right] \ln x\,\lambda^1(dx)\\ &= \int \mathbb{1}_{(1,\infty)}(x) e^{-x} \ln x\,\lambda^1(dx).\end{aligned}$$

That $e^{-x}\ln x$ is integrable in $(1,\infty)$ follows easily from the estimates

$$e^{-x} \leqslant C_N x^{-N} \quad \text{and} \quad \ln x \leqslant x$$

which hold for all $x \geqslant 1$ and $N \in \mathbb{N}$.

(ii) Note that $x \mapsto \ln x$ is continuous and bounded in $[\epsilon, 1]$, thus Riemann integrable. It is easy to see that $x\ln x - x$ is a primitive for $\ln x$. The improper Riemann integral

$$\int_0^1 \ln x\,dx = \lim_{\epsilon\to 0}[x\ln x - x]_\epsilon^1 = -1$$

exists and, since $\ln x$ is negative throughout $(0,1)$, improper Riemann and Lebesgue integrals coincide. Thus, $\ln x \in L^1(dx, (0,1))$.

Therefore,

$$\left|\left(1 - \frac{x}{k}\right)^k \ln x\right| \leqslant |\ln x|, \quad \forall\, x \in (0,1)$$

is uniformly dominated by an integrable function and we can use dominated convergence to get

$$\begin{aligned}\lim_k \int_{(0,1)} \left(1 - \frac{x}{k}\right)^k \ln x\,dx &= \int_{(0,1)} \lim_k \left(1 - \frac{x}{k}\right)^k \ln x\,dx\\ &= \int_{(0,1)} e^{-x} \ln x\,dx\end{aligned}$$

Problem 11.13 Fix throughout $(a,b) \subset (0,\infty)$ and take $x \in (a,b)$. Let us remark that, just as in Problem 11.8, we prove that

$$\int_{(0,1)} t^{-\delta}\,dt < \infty \quad \forall\,\delta < 1 \qquad \text{and} \qquad \int_{(1,\infty)} t^{-\delta}\,dt < \infty \quad \forall\,\delta > 1.$$

(i) That the integrand function $x \mapsto \gamma(t,x)$ is continuous on (a,b) is clear. It is therefore enough to find an integrable dominating function. We have

$$e^{-t}t^{x-1} \leqslant t^{a-1} \qquad \forall\, t \in (0,1), \quad x \in (a,b)$$

which is clearly integrable on $(0,1)$ and

$$e^{-t}t^{x-1} \leqslant M_{a,b}\, t^{-2} \qquad \forall\, t \geqslant 1, \quad x \in (a,b)$$

where we used that $t^{\rho}e^{-t}$, $\rho > 0$, is continuous and $\lim_{t\to\infty} t^{\rho}e^{-t} = 0$ to find $M_{a,b}$. This function is integrable over $[1,\infty)$. Both estimates together give the wanted integrable dominating function. The Continuity lemma (Theorem 11.4) applies. The well-definedness of $\Gamma(x)$ comes for free as a by-product of the existence of the dominating function.

(ii) *Induction Hypothesis:* $\Gamma^{(m)}$ exists and is of the form as claimed in the statement of the problem.

Induction Start $m = 1$: We have to show that $\Gamma(x)$ is differentiable. We want to use the Differentiability lemma, Theorem 11.5. For this we remark first of all, that the integrand function $x \mapsto \gamma(t,x)$ is differentiable on (a,b) and that

$$\partial_x \gamma(t,x) = \partial_x e^{-t}\, t^{x-1} = e^{-t}\, t^{x-1} \log t.$$

We have now to find a uniform (for $x \in (a,b)$) integrable dominating function for $|\partial_x \gamma(t,x)|$. Since $\log t \leqslant t$ for all $t > 0$ (the logarithm is a concave function!),

$$\begin{aligned}\left|e^{-t}\, t^{x-1} \log t\right| &= e^{-t}\, t^{x-1} \log t \\ &\leqslant e^{-t}t^{x} \;\leqslant\; e^{-t}t^{b} \;\leqslant\; C_b\, t^{-2} \qquad \forall\, t \geqslant 1, \quad x \in (a,b)\end{aligned}$$

(use for the last step the argument used in part (i) of this problem). Moreover,

$$\begin{aligned}\left|e^{-t}\, t^{x-1} \log t\right| &\leqslant t^{a-1}\, |\log t| \\ &= t^{a-1} \log \frac{1}{t} \;\leqslant\; C_a\, t^{-1/2} \qquad \forall\, t \in (0,1), \quad x \in (a,b)\end{aligned}$$

where we used the fact that $\lim_{t\to 0} t^{\rho} \log \frac{1}{t} = 0$ which is easily seen by the substitution $t = e^{-u}$ and $u \to \infty$ and the continuity of the function $t^{\rho} \log \frac{1}{t}$.

Both estimates together furnish an integrable dominating function, so the Differentiability lemma applies and shows that

$$\Gamma'(x) = \int_{(0,\infty)} \partial_x \gamma(t,x)\, dt = \int_{(0,\infty)} e^{-t}\, t^{x-1} \log t\, dt = \Gamma^{(1)}(x).$$

Induction Step $m \rightsquigarrow m+1$: Set $\gamma^{(m)}(t,x) = e^{-t}\, t^{x-1} (\log t)^m$. We want to apply the Differentiability Lemma to $\Gamma^{(m)}(x)$. With very much the same arguments as in the induction start we find that $\gamma^{(m+1)}(t,x) = \partial_x \gamma^{(m)}(t,x)$ exists (obvious) and satisfies the following bounds

$$\begin{aligned}
\left|e^{-t}\, t^{x-1}\, (\log t)^{m+1}\right| &= e^{-t}\, t^{x-1}\, (\log t)^{m+1} \\
&\leqslant e^{-t} t^{x+m} \\
&\leqslant e^{-t} t^{b+m} \\
&\leqslant C_{b,m}\, t^{-2} \qquad \forall\, t \geqslant 1,\ \ x \in (a,b) \\
\left|e^{-t}\, t^{x-1}\, (\log t)^{m+1}\right| &\leqslant t^{a-1}\, |\log t|^m \\
&= t^{a-1} \left(\log \frac{1}{t}\right)^{m+1} \\
&\leqslant C_{a,m}\, t^{-1/2} \qquad \forall\, t \in (0,1),\ \ x \in (a,b)
\end{aligned}$$

and the Differentiability lemma applies completing the induction step.

(iii) Using a combination of Beppo-Levi (indicated by 'B-L'), Riemann=Lebesgue (if the Riemann integral over an interval exists) and integration by parts (for the Riemann integral, indicated by 'I-by-P') techniques we get

$$\begin{aligned}
x\Gamma(x) &= \lim_{n\to\infty} \int_{(1/n,n)} e^{-t}\, x t^{x-1}\, dt && \text{B-L} \\
&= \lim_{n\to\infty} (R) \int_{1/n}^{n} e^{-t}\, \partial_t t^x\, dt \\
&= \lim_{n\to\infty} \left[e^{-t}\, t^x\right]_{t=1/n}^{n} - \lim_{n\to\infty} (R) \int_{1/n}^{n} \partial_t e^{-t}\, t^x\, dt && \text{I-by-P} \\
&= \lim_{n\to\infty} (R) \int_{1/n}^{n} e^{-t}\, t^{(x+1)-1}\, dt \\
&= \lim_{n\to\infty} \int_{(1/n,n)} e^{-t}\, t^{(x+1)-1}\, dt \\
&= \int_{(0,\infty)} e^{-t}\, t^{(x+1)-1}\, dt && \text{B-L} \\
&= \Gamma(x+1).
\end{aligned}$$

Problem 11.14 Fix $(a,b) \subset (0,1)$ and let always $u \in (a,b)$. We have for $x \geqslant 0$ and $L \in \mathbb{N}_0$

$$\begin{aligned}
|x^L f(u,x)| &= |x|^L \left|\frac{e^{ux}}{e^x+1}\right| \\
&= x^L\, \frac{e^{ux}}{e^x+1} \\
&\leqslant x^L\, \frac{e^{ux}}{e^x} \\
&= x^L\, e^{(u-1)x} \\
&\leqslant \mathbb{1}_{[0,1]}(x) + M_{a,b} \mathbb{1}_{(1,\infty)}(x)\, x^{-2}
\end{aligned}$$

where we used that $u-1<0$, the continuity and boundedness of $x^\rho e^{-ax}$ for $x \in [1,\infty)$ and $\rho \geqslant 0$. If $x \leqslant 0$ we get

$$|x^L f(u,x)| = |x|^L \left|\frac{e^{ux}}{e^x+1}\right|$$

$$= |x|^L e^{-u|x|}$$
$$\leqslant \mathbb{1}_{[-1,0]}(x) + N_{a,b}\mathbb{1}_{(-\infty,1)}(x)\,|x|^{-2}.$$

Both inequalities give dominating functions which are integrable; therefore, the integral $\int_{\mathbb{R}} x^L f(u,x)\,dx$ exists.

To see m-fold differentiability, we use the Differentiability lemma (Theorem 11.5) m-times. Formally, we have to use induction. Let us only make the induction step (the start is very similar!). For this, observe that

$$\partial_u^m (x^n f(u,x)) = \partial_u^m \frac{x^n e^{ux}}{e^x+1} = \frac{x^{n+m} e^{ux}}{e^x+1}$$

but, as we have seen in the first step with $L = n+m$, this is uniformly bounded by an integrable function. Therefore, the Differentiability lemma applies and shows that

$$\partial_u^m \int_{\mathbb{R}} x^n\, f(u,x)\,dx = \int_{\mathbb{R}} x^n\, \partial_u^m f(u,x)\,dx = \int_{\mathbb{R}} x^{n+m}\, f(u,x)\,dx.$$

Problem 11.15 Note the *misprint* in this problem: the random variable X should be *positive.*

(i) Since

$$\left|\frac{d^m}{dt^m}\, e^{-tX}\right| = \left|X^m e^{-tX}\right| \leqslant X^m$$

m applications of the differentiability lemma, Theorem 11.5, show that $\phi_X^{(m)}(0+)$ exists and that

$$\phi_X^{(m)}(0+) = (-1)^m \int X^m\,dP.$$

(ii) Using the exponential series we find that

$$e^{-tX} - \sum_{k=0}^{m} X^k\,\frac{(-1)^k t^k}{k!} = \sum_{k=m+1}^{\infty} X^k\,\frac{(-1)^k t^k}{k!}$$
$$= t^{m+1} \sum_{j=0}^{\infty} X^{m+1+j}\,\frac{(-1)^{m+1+j} t^j}{(m+1+j)!}\,.$$

Since the left-hand side has a finite P-integral, so has the right, i.e.

$$\int \Big(\sum_{j=0}^{\infty} X^{m+1+j}\,\frac{(-1)^{m+1+j} t^j}{(m+1+j)!}\Big)\,dP \qquad \text{converges}$$

and we see that

$$\int \Big(e^{-tX} - \sum_{k=0}^{m} X^k\,\frac{(-1)^k t^k}{k!}\Big)\,dP = o(t^m)$$

as $t \to 0$.

(iii) We show, by induction in m, that

$$\left|e^{-u} - \sum_{k=0}^{m-1} \frac{(-u)^k}{k!}\right| \leqslant \frac{u^m}{m!} \qquad \forall\, u \geqslant 0. \tag{*}$$

Because of the elementary inequality

$$|e^{-u}-1|\leqslant u \qquad \forall\, u\geqslant 0$$

the start of the induction $m=1$ is clear. For the induction step $m\to m+1$ we note that

$$\begin{aligned}\left|e^{-u}-\sum_{k=0}^{m}\frac{(-u)^k}{k!}\right| &= \left|\int_0^u\Big(e^{-y}-\sum_{k=0}^{m-1}\frac{(-y)^k}{k!}\Big)\,dy\right|\\ &\leqslant \int_0^u\left|e^{-y}-\sum_{k=0}^{m-1}\frac{(-y)^k}{k!}\right|dy\\ &\overset{(*)}{\leqslant} \int_0^u \frac{y^m}{m!}\,dy\\ &= \frac{u^{m+1}}{(m+1)!},\end{aligned}$$

and the claim follows.

Setting $x=tX$ in (*), we find by integration that

$$\pm\Big(\int e^{-tX}-\sum_{k=0}^{m-1}(-1)^k t^k\,\frac{\int X^k\,dP}{k!}\Big)\leqslant \frac{t^m\int X^m\,dP}{m!}.$$

(iv) If t is in the radius of convergence of the power series, we know that

$$\lim_{m\to\infty}\frac{|t|^m\int X^m\,dP}{m!}=0$$

which, when combined with (iii), proves that

$$\phi_X(t)=\lim_{m\to\infty}\sum_{k=0}^{m-1}(-1)^k t^k\,\frac{\int X^k\,dP}{k!}.$$

Problem 11.16 (i) Wrong, u is NOT continuous on the irrational numbers. To see this, just take a sequence of rationals $q_j\in\mathbb{Q}\cap[0,1]$ approximating $p\in[0,1]\smallsetminus\mathbb{Q}$. Then

$$\lim_j u(q_j)=1\neq 0=u(p)=u(\lim_j q_j).$$

(ii) True. Mind that v is not continuous at 0, but $\{n^{-1}, n\in\mathbb{N}\}\cup\{0\}$ is still countable.

(iii) True. The points where u and v are not 0 (that is: where they are 1) are countable sets, hence measurable and also Lebesgue null sets. This shows that u,v are measurable and almost everywhere 0, hence $\int u\,d\lambda=0=\int v\,d\lambda$.

(iv) True. Since $\mathbb{Q}\cap[0,1]$ as well as $[0,1]\smallsetminus\mathbb{Q}$ are dense subsets of $[0,1]$, ALL lower resp. upper Darboux sums are always

$$S_\pi[u]\equiv 0 \ \text{ resp. } \ S^\pi[u]\equiv 1$$

(for any finite partition π of $[0,1]$). Thus upper and lower integrals of u have the value 0 resp. 1 and it follows that u cannot be Riemann integrable.

Problem 11.17 Note that every function which has finitely many discontinuities is Riemann integrable. Thus, if $\{q_j\}_{j\in\mathbb{N}}$ is an enumeration of $\mathbb{Q}$, the functions $u_j(x) := \mathbb{1}_{\{q_1,q_2,\dots,q_j\}}(x)$ are Riemann integrable (with Riemann integral 0) while their increasing limit $u_\infty = \mathbb{1}_\mathbb{Q}$ is not Riemann integrable.

Problem 11.18 Of course we have to assume that u is Borel measurable! By assumption we know that $u_j := u\mathbb{1}_{[0,j]}$ is (properly) Riemann integrable, hence Lebesgue integrable and

$$\int_{[0,j]} u\,d\lambda = \int_{[0,j]} u_j\,d\lambda = (\mathrm{R})\int_0^j u(x)\,dx \xrightarrow{j\to\infty} \int_0^\infty u(x)\,dx.$$

The last limit exists because of improper Riemann integrability. Moreover, this limit is an increasing limit, i.e. a 'sup'. Since $0 \leqslant u_j \uparrow u$ we can invoke Beppo Levi's theorem and get

$$\int u\,d\lambda = \sup_j \int u_j\,d\lambda = \int_0^\infty u(x)\,dx < \infty$$

proving Lebesgue integrability.

Problem 11.19 Observe that $x^2 = k\pi \iff x = \sqrt{k\pi},\ x \geqslant 0,\ k \in \mathbb{N}_0$. Thus, Since $\sin x^2$ is continuous, it is on every bounded interval Riemann integrable. By a change of variables, $y = x^2$, we get

$$\int_{\sqrt{a}}^{\sqrt{b}} |\sin(x^2)|\,dx = \int_a^b |\sin y|\,\frac{dy}{2\sqrt{y}} = \int_a^b \frac{|\sin y|}{2\sqrt{y}}\,dy$$

which means that for $a = a_k = k\pi$ and $b = b_k = (k+1)\pi = a_{k+1}$ the values $\int_{\sqrt{a_k}}^{\sqrt{a_{k+1}}} |\sin(x^2)|\,dx$ are a decreasing sequence with limit 0. Since on $\left[\sqrt{a_k}, \sqrt{a_{k+1}}\right]$ the function $\sin x^2$ has only one sign (and alternates its sign from interval to interval), we can use Leibniz' convergence criterion to see that the series

$$\sum_k \int_{\sqrt{a_k}}^{\sqrt{a_{k+1}}} \sin(x^2)\,dx \tag{*}$$

converges, hence the improper integral exists.

The function $\cos x^2$ can be treated similarly. Alternatively, we remark that $\sin x^2 = \cos(x^2 - \pi/2)$.

The functions are not Lebesgue integrable. Either we show that the series (*) does not converge absolutely, or we argue as follows:

$\sin x^2 = \cos(x^2 - \pi/2)$ shows that $\int |\sin x^2|\,dx$ and $\int |\cos x^2|\,dx$ either both converge or diverge. If they would converge (this is equivalent to Lebesgue integrability...) we would find because of $\sin^2 + \cos^2 \equiv 1$ and $|\sin|, |\cos| \leqslant 1$,

$$\begin{aligned}
\infty = \int_0^\infty 1\,dx &= \int_0^\infty \left[(\sin x^2)^2 + (\cos x^2)^2\right] dx \\
&= \int_0^\infty (\sin x^2)^2\,dx + \int_0^\infty (\cos x^2)^2\,dx \\
&\leqslant \int_0^\infty |\sin x^2|\,dx + \int_0^\infty |\cos x^2|\,dx \ < \ \infty,
\end{aligned}$$

which is a contradiction.

Problem 11.20 Let $r < s$ and, without loss of generality, $a \leqslant b$. A change of variables yields

$$\begin{aligned}\int_r^s \frac{f(bx)-f(ax)}{x}\,dx &= \int_r^s \frac{f(bx)}{x}\,dx - \int_r^s \frac{f(ax)}{x}\,dx\\ &= \int_{br}^{bs} \frac{f(y)}{y}\,dy - \int_{ar}^{as} \frac{f(y)}{y}\,dy\\ &= \int_{as}^{bs} \frac{f(y)}{y}\,dy - \int_{ar}^{br} \frac{f(y)}{y}\,dy\end{aligned}$$

Using the mean value theorem for integrals, E.12, we get

$$\begin{aligned}\int_r^s \frac{f(bx)-f(ax)}{x}\,dx &= f(\xi_s)\int_{as}^{bs}\frac{1}{y}\,dy - f(\xi_r)\int_{ar}^{br}\frac{1}{y}\,dy\\ &= f(\xi_s)\ln\tfrac{b}{a} - f(\xi_r)\ln\tfrac{b}{a}.\end{aligned}$$

Since $\xi_s \in (as, bs)$ and $\xi_r \in (ar, br)$, we find that $\xi_s \xrightarrow{s\to\infty} \infty$ and $\xi_r \xrightarrow{r\to 0} 0$ which means that

$$\int_r^s \frac{f(bx)-f(ax)}{x}\,dx = \left[f(\xi_s) - f(\xi_r)\right]\ln\tfrac{b}{a} \xrightarrow[r\to 0]{s\to\infty} (M-m)\ln\tfrac{b}{a}.$$

Problem 11.21 (i) The function $x \mapsto x\ln x$ is bounded and continuous in $[0,1]$, hence Riemann integrable. Since in this case Riemann and Lebesgue integrals coincide, we may use Riemann's integral and the usual rules for integration. Thus, changing variables according to $x = e^{-t}$, $dx = -e^{-t}\,dt$ and then $s = (k+1)t$, $ds = (k+1)\,ds$ we find,

$$\begin{aligned}\int_0^1 (x\ln x)^k\,dx &= \int_0^\infty \left[e^{-t}(-t)\right]^k e^{-t}\,dt\\ &= (-1)^k \int_0^\infty t^k e^{-t(k+1)}\,dt\\ &= (-1)^k \int_0^\infty \Big(\frac{s}{k+1}\Big)^k e^{-s}\,\frac{ds}{k+1}\\ &= (-1)^k \Big(\frac{1}{k+1}\Big)^{k+1} \int_0^\infty s^{(k+1)-1} e^{-s}\,ds\\ &= (-1)^k \Big(\frac{1}{k+1}\Big)^{k+1} \Gamma(k+1).\end{aligned}$$

(ii) Following the hint we write

$$x^{-x} = e^{-x\ln x} = \sum_{k=0}^\infty (-1)^k \frac{(x\ln x)^k}{k!}.$$

Since for $x \in (0,1)$ the terms under the sum are all positive, we can use Beppo Levi's theorem and the formula $\Gamma(k+1) = k!$ to get

$$\begin{aligned}\int_{(0,1)} x^{-x}\,dx &= \sum_{k=0}^\infty (-1)^k \frac{1}{k!}\int_{(0,1)} (x\ln x)^k\,dx\\ &= \sum_{k=0}^\infty (-1)^k \frac{1}{k!}(-1)^k \Big(\frac{1}{k+1}\Big)^{k+1}\Gamma(k+1)\\ &= \sum_{k=0}^\infty \Big(\frac{1}{k+1}\Big)^{k+1}\\ &= \sum_{n=1}^\infty \Big(\frac{1}{n}\Big)^n.\end{aligned}$$

12 The function spaces $\mathcal{L}^p$, $1 \leqslant p \leqslant \infty$. Solutions to Problems 12.1–12.22

Problem 12.1 (i) We use Hölder's inequality for $r, s \in (1,\infty)$ and $\frac{1}{s} + \frac{1}{t} = 1$ to get

$$\begin{aligned}\|u\|_q^q = \int |u|^q \, d\mu &= \int |u|^q \cdot \mathbb{1} \, d\mu \\ &\leqslant \Big(\int |u|^{qr} \, d\mu\Big)^{1/r} \cdot \Big(\int \mathbb{1}^s \, d\mu\Big)^{1/s} \\ &= \Big(\int |u|^{qr} \, d\mu\Big)^{1/r} \cdot (\mu(X))^{1/s}.\end{aligned}$$

Now let us choose r and s. We take

$$r = \frac{p}{q} > 1 \implies \frac{1}{r} = \frac{q}{p} \qquad \text{and} \qquad \frac{1}{s} = 1 - \frac{1}{r} = 1 - \frac{q}{p},$$

hence

$$\begin{aligned}\|u\|_q &= \Big(\int |u|^p \, d\mu\Big)^{q/p \cdot 1/q} \cdot (\mu(X))^{(1-q/p)(1/q)} \\ &= \Big(\int |u|^p \, d\mu\Big)^{q/p \cdot 1/q} \cdot (\mu(X))^{1/q-1/p} \\ &= \|u\|_p \cdot (\mu(X))^{1/q-1/p}.\end{aligned}$$

(ii) If $u \in \mathcal{L}^p$ we know that u is measurable and $\|u\|_p < \infty$. The inequality in (i) then shows that

$$\|u\|_q \leqslant \text{const} \cdot \|u\|_p < \infty,$$

hence $u \in \mathcal{L}^q$. This gives $\mathcal{L}^p \subset \mathcal{L}^q$. The inclusion $\mathcal{L}^q \subset \mathcal{L}^1$ follows by taking $p \leadsto q$, $q \leadsto 1$.

Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^p$ be a Cauchy sequence, i.e. $\lim_{m,n\to\infty} \|u_n - u_m\|_p = 0$. Since by the inequality in (i) also

$$\lim_{m,n\to\infty} \|u_n - u_m\|_q \leqslant \mu(X)^{1/q-1/p} \lim_{m,n\to\infty} \|u_n - u_m\|_p = 0$$

we get that $(u_n)_{n\in\mathbb{N}} \subset \mathcal{L}^q$ is also a Cauchy sequence in $\mathcal{L}^q$.

(iii) No, the assertion breaks down completely if the measure μ has infinite mass. Here is an example: μ = Lebesgue measure on $(1,\infty)$. Then the function $f(x) = \frac{1}{x}$ is not integrable over $[1,\infty)$, but $f^2(x) = \frac{1}{x^2}$ is. In other words: $f \notin \mathcal{L}^1(1,\infty)$ but $f \in \mathcal{L}^2(1,\infty)$, hence $\mathcal{L}^2(1,\infty) \not\subset \mathcal{L}^1(1,\infty)$. (Playing around with different exponents shows that the assertion also fails for other $p, q \geqslant 1$....).

Problem 12.2 This is going to be a bit messy and rather than showing the 'streamlined' solution we indicate how one could find out the numbers oneself. Now let λ be some number in $(0,1)$ and let α,β be conjugate indices: $\frac{1}{\alpha}+\frac{1}{\beta}=1$ where $\alpha,\beta\in(1,\infty)$. Then by the Hölder inequality

$$\begin{aligned}\int |u|^r\,d\mu &= \int |u|^{r\lambda}|u|^{r(1-\lambda)}\,d\mu\\ &\leqslant \left(\int |u|^{r\lambda\alpha}\,d\mu\right)^{\frac{1}{\alpha}}\left(\int |u|^{r(1-\lambda)\beta}\,d\mu\right)^{\frac{1}{\beta}}\\ &= \left(\int |u|^{r\lambda\alpha}\,d\mu\right)^{\frac{r\lambda}{r\lambda\alpha}}\left(\int |u|^{r(1-\lambda)\beta}\,d\mu\right)^{\frac{r(1-\lambda)}{r(1-\lambda)\beta}}.\end{aligned}$$

Taking rth roots on both sides yields

$$\begin{aligned}\|u\|_r &\leqslant \left(\int |u|^{r\lambda\alpha}\,d\mu\right)^{\frac{\lambda}{r\lambda\alpha}}\left(\int |u|^{r(1-\lambda)\beta}\,d\mu\right)^{\frac{(1-\lambda)}{r(1-\lambda)\beta}}\\ &= \|u\|_{r\lambda\alpha}^{\lambda}\|u\|_{r(1-\lambda)\beta}^{1-\lambda}.\end{aligned}$$

This leads to the following system of equations:

$$p = r\lambda\alpha q \qquad\qquad = r(1-\lambda)\beta 1 = \frac{1}{\alpha}+\frac{t}{\beta}$$

with unknown quantities α,β,λ. Solving it yields

$$\lambda = \frac{\frac{1}{r}-\frac{1}{q}}{\frac{1}{p}-\frac{1}{q}},\qquad \alpha = \frac{q-p}{q-r}\qquad \beta = \frac{q-p}{r-p}.$$

Problem 12.3 $v\in\mathcal{L}^\infty(\mu)$ means that $|v(x)|\leqslant(\|v\|_\infty+\epsilon)$ for all $x\in N=N_\epsilon$ with $\mu(N)=0$. Using in step $*$ below Theorem 10.9, we get

$$\begin{aligned}\int uv\,d\mu &\leqslant \int |u||v|\,d\mu\\ &\overset{*}{=} \int_{N^c} |u||v|\,d\mu\\ &\leqslant \int_{N^c} |u|\big(\|v\|_\infty+\epsilon\big)\,d\mu\\ &= \big(\|v\|_\infty+\epsilon\big)\int_{N^c}|u|\,d\mu\\ &\leqslant \big(\|v\|_\infty+\epsilon\big)\int |u|\,d\mu\end{aligned}$$

and since the left-hand side does not depend on $\epsilon>0$, we can let $\epsilon\to 0$ and find

$$\int uv\,d\mu \leqslant \left|\int uv\,d\mu\right| \leqslant \int |uv|\,d\mu \leqslant \big(\|v\|_\infty+\epsilon\big)\|u\|_1 \xrightarrow{\epsilon\to 0} \|v\|_\infty\|u\|_1.$$

Problem 12.4 Proof by induction in N.

Start $N=2$*:* this is just Hölder's inequality.

Hypothesis: the generalized Hölder inequality holds for some $N\geqslant 2$.

Step $N \rightsquigarrow N+1$:. Let $u_1, \dots, u_N, w$ be $N+1$ functions and let $p_1, \dots, p_N, q > 1$ be such that $p_1^{-1} + p_2^{-1} + \dots + p_N^{-1} + q^{-1} = 1$. Set $p^{-1} := p_1^{-1} + p_2^{-1} + \dots + p_N^{-1}$. Then, by the ordinary Hölder inequality,

$$\begin{aligned}\int |u_1 \cdot u_2 \cdot \ldots \cdot u_N \cdot w| \, d\mu &\leqslant \left(\int |u_1 \cdot u_2 \cdot \ldots \cdot u_N|^p \, d\mu \right)^{1/p} \|u\|_q \\ &= \left(\int |u_1|^p \cdot |u_2|^p \cdot \ldots \cdot |u_N|^p \, d\mu \right)^{1/p} \|u\|_q\end{aligned}$$

Now use the induction hypothesis which allows us to apply the generalized Hölder inequality for N (!) factors $\lambda_j := p/p_j$, and thus $\sum_{j=1}^N \lambda_j^{-1} = p/p = 1$, to the first factor to get

$$\begin{aligned}\int |u_1 \cdot u_2 \cdot \ldots \cdot u_N \cdot w| \, d\mu &= \left(\int |u_1|^p \cdot |u_2|^p \cdot \ldots \cdot |u_N|^p \, d\mu \right)^{1/p} \|u\|_q \\ &\leqslant \|u\|_{p_1} \cdot \|u\|_{p_2} \cdot \ldots \cdot \|u\|_{p_N} \|u\|_q.\end{aligned}$$

Problem 12.5 Draw a picture similar to the one used in the proof of Lemma 12.1 (note that the increasing function need not be convex or concave....). Without loss of generality we can assume that $A, B > 0$ are such that $\phi(A) \geqslant B$ which is equivalent to $A \geqslant \psi(B)$ since ϕ and ψ are inverses. Thus,

$$AB = \int_0^B \psi(\eta) \, d\eta + \int_0^{\psi(B)} \phi(\xi) \, d\xi + \int_{\psi(B)}^A B \, d\xi.$$

Using the fact that ϕ increases, we get that

$$\phi(\psi(B)) = B \implies \phi(C) \geqslant B \quad \forall C \geqslant \psi(B)$$

and we conclude that

$$\begin{aligned}AB &= \int_0^B \psi(\eta) \, d\eta + \int_0^{\psi(B)} \phi(\xi) \, d\xi + \int_{\psi(B)}^A B \, d\xi \\ &\leqslant \int_0^B \psi(\eta) \, d\eta + \int_0^{\psi(B)} \phi(\xi) \, d\xi + \int_{\psi(B)}^A \phi(\xi) \, d\xi \\ &= \int_0^B \psi(\eta) \, d\eta + \int_0^A \phi(\xi) \, d\xi \\ &= \Psi(B) + \Phi(A).\end{aligned}$$

Problem 12.6 Let us show first of all that $\mathcal{L}^p\text{-}\lim_{k\to\infty} u_k = u$. This follows immediately from $\lim_{k\to\infty} \|u - u_k\|_p = 0$ since the series $\sum_{k=1}^\infty \|u - u_k\|_p$ converges.

Therefore, we can find a subsequence $(u_{k(j)})_{j\in\mathbb{N}}$ such that

$$\lim_{j\to\infty} u_{k(j)}(x) = u(x) \quad \text{almost everywhere.}$$

Now we want to show that u is the a.e. limit of the original sequence. For this we mimic the trick from the Riesz-Fischer theorem 12.7 and show that the series

$$\sum_{j=0}^\infty (u_{j+1} - u_j) = \lim_{K\to\infty} \sum_{j=0}^K (u_{j+1} - u_j) = \lim_{K\to\infty} u_K$$

(again we agree on $u_0 := 0$ for notational convenience) makes sense. So let us employ Lemma 12.6 used in the proof of the Riesz-Fischer theorem to get

$$\begin{aligned}\Big\|\sum_{j=0}^{\infty}(u_{j+1}-u_j)\Big\|_p &\leqslant \Big\|\sum_{j=0}^{\infty}|u_{j+1}-u_j|\Big\|_p\\ &\leqslant \sum_{j=0}^{\infty}\|u_{j+1}-u_j\|_p\\ &\leqslant \sum_{j=0}^{\infty}\left(\|u_{j+1}-u\|_p+\|u-u_j\|_p\right)\\ &< \infty\end{aligned}$$

where we used Minkowski's inequality, the function u from above and the fact that $\sum_{j=1}^{\infty}\|u_j-u\|_p < \infty$ along with $\|u_1\|_p < \infty$. This shows that $\lim_{K\to\infty} u_K(x) = \sum_{j=0}^{\infty}(u_{j+1}(x)-u_j(x))$ exists almost everywhere.

We still have to show that $\lim_{K\to\infty} u_K(x) = u(x)$. For this we remark that a subsequence has necessarily the same limit as the original sequence—whenever both have limits, of course. But then,

$$u(x) = \lim_{j\to\infty} u_{k(j)}(x) = \lim_{k\to\infty} u_k(x) = \sum_{j=0}^{\infty}(u_{j+1}(x)-u_j(x))$$

and the claim follows.

Problem 12.7 That for every fixed x the sequence

$$u_n(x) := n\mathbb{1}_{(0,1/n)}(x) \xrightarrow{n\to\infty} 0$$

is obvious. On the other hand, for any subsequence $(u_{n(j)})_j$ we have

$$\int |u_{n(j)}|^p\,d\lambda = n(j)^p\,\frac{1}{n(j)} = n(j)^{p-1} \xrightarrow{j\to\infty} c$$

with $c = 1$ in case $p = 1$ and $c = \infty$ if $p > 1$. This shows that the $\mathcal{L}^p$-limit of this subsequence—let us call it w if it exists at all—cannot be (not even a.e.) $u = 0$.

On the other hand, we know that a sub-subsequence $(\tilde u_{k(j)})_j$ of $(u_{k(j)})_j$ converges pointwise almost everywhere to the $\mathcal{L}^p$-limit:

$$\lim_j \tilde u_{k(j)}(x) = w(x).$$

Since the full sequence $\lim_n u_n(x) = u(x) = 0$ has a limit, this shows that the sub-subsequence limit $w(x) = 0$ almost everywhere—a contradiction. Thus, w does not exist in the first place.

Problem 12.8 Using Minkowski's and Hölder's inequalities we find for all $\epsilon > 0$

$$\|u_k v_k - uv\|_1 = \|u_k v_k - u_k v + u_k v - uv\|$$

$$\leqslant \|u_k \cdot (v_k - v)\| + \|(u_k - u)v\|$$
$$\leqslant \|u_k\|_p \|v_k - v\|_q + \|u_k - u\|_p \|v\|_q$$
$$\leqslant (M + \|v\|_q)\epsilon$$

for all $n \geqslant N_\epsilon$. We used here that the sequence $(\|u_k\|_p)_{k\in\mathbb{N}}$ is bounded. Indeed, by Minkowski's inequality

$$\|u_k\|_p = \|u_k - u\|_p + \|u\|_p \leqslant \epsilon + \|u\|_p =: M.$$

Problem 12.9 We use the simple identity

$$\begin{aligned} \|u_n - u_m\|_2^2 &= \int (u_n - u_m)^2 \, d\mu \\ &= \int (u_n^2 - 2u_n u_m + u_m) \, d\mu \qquad (*) \\ &= \|u_n\|_2^2 + \|u_m\|_2^2 - 2\int u_n u_m \, d\mu. \end{aligned}$$

Case 1: $u_n \to u$ in $\mathcal{L}^2$. This means that $(u_n)_{n\in\mathbb{N}}$ is an $\mathcal{L}^2$ Cauchy sequence, i.e. that $\lim_{m,n\to\infty} \|u_n - u_m\|_2^2 = 0$. On the other hand, we get from the lower triangle inequality for norms

$$\lim_{n\to\infty} |\|u_n\|_2 - \|u\|_2| \leqslant \lim_{n\to\infty} \|u_n - u\|_2 = 0$$

so that also $\lim_{n\to\infty} \|u_n\|_2^2 = \lim_{m\to\infty} \|u_m\|_2^2 = \|u\|_2^2$. Using (*) we find

$$\begin{aligned} 2\int u_n u_m \, d\mu &= \|u_n\|_2^2 + \|u_m\|_2^2 - \|u_n - u_m\|_2^2 \\ &\xrightarrow{n,m\to\infty} \|u\|_2^2 + \|u\|_2^2 - 0 \\ &= 2\|u\|_2^2. \end{aligned}$$

Case 2: Assume that $\lim_{n,m\to\infty} \int u_n u_m \, d\mu = c$ for some number $c \in \mathbb{R}$. By the very definition of this double limit, i.e.

$$\forall\, \epsilon > 0 \quad \exists\, N_\epsilon \in \mathbb{N} \quad : \quad \left| \int u_n u_m \, d\mu - c \right| < \epsilon \quad \forall n, m \geqslant N_\epsilon,$$

we see that $\lim_{n\to\infty} \int u_n u_n \, d\mu = c = \lim_{m\to\infty} \int u_m u_m \, d\mu$ hold (with the same c!). Therefore, again by (*), we get

$$\begin{aligned} \|u_n - u_m\|_2^2 &= \|u_n\|_2^2 + \|u_m\|_2^2 - 2\int u_n u_m \, d\mu \\ &\xrightarrow{n,m\to\infty} c + c - 2c \;=\; 0, \end{aligned}$$

i.e. $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{L}^2$ and has, by the completeness of this space, a limit.

Problem 12.10 Use the exponential series to conclude from the positivity of h and $u(x)$ that

$$\exp(hu) = \sum_{j=0}^{\infty} \frac{h^j u^j}{j!} \geqslant \frac{h^N}{N!} u^N.$$

Integrating this gives

$$\frac{h^N}{N!}\int u^N\,d\mu \leqslant \int \exp(hu)\,d\mu < \infty$$

and we find that $u \in \mathcal{L}^N$. Since μ is a finite measure we know from Problem 12.1 that for $N > p$ we have $\mathcal{L}^N \subset \mathcal{L}^p$.

Problem 12.11 (i) We have to show that $|u_n(x)|^p := n^{p\alpha}(x+n)^{-p\beta}$ has finite integral—measurability is clear since u_n is continuous. Since $n^{p\alpha}$ is a constant, we have only to show that $(x+n)^{-p\beta}$ is in $\mathcal{L}^1$. Set $\gamma := p\beta > 1$. Then we get from a Beppo-Levi and a domination argument

$$\begin{aligned}\int_{(0,\infty)}(x+n)^{-\gamma}\,\lambda(dx) &\leqslant \int_{(0,\infty)}(x+1)^{-\gamma}\,\lambda(dx)\\ &\leqslant \int_{(0,1)}1\,\lambda(dx) + \int_{(1,\infty)}(x+1)^{-\gamma}\,\lambda(dx)\\ &\leqslant 1 + \lim_{k\to\infty}\int_{(1,k)}x^{-\gamma}\,\lambda(dx).\end{aligned}$$

Now using that Riemann=Lebesgue on intervals where the Riemann integral exists, we get

$$\begin{aligned}\lim_{k\to\infty}\int_{(1,k)}x^{-\gamma}\,\lambda(dx) &= \lim_{k\to\infty}\int_1^k x^{-\gamma}\,dx\\ &= \lim_{k\to\infty}\left[(1-\gamma)^{-1}x^{1-\gamma}\right]_1^k\\ &= (1-\gamma)^{-1}\lim_{k\to\infty}\left(k^{1-\gamma}-1\right)\\ &= (\gamma-1)^{-1} \;<\; \infty\end{aligned}$$

which shows that the integral is finite.

(ii) We have to show that $|v_n(x)|^q := n^{q\gamma}e^{-qnx}$ is in $\mathcal{L}^1$—again measurability is inferred from continuity. Since $n^{q\gamma}$ is a constant, it is enough to show that e^{-qnx} is integrable. Set $\delta = qn$. Since

$$\lim_{x\to\infty}(\delta x)^2 e^{-\delta x} = 0 \qquad\text{and}\qquad e^{-\delta x}\leqslant 1 \quad \forall\, x\geqslant 0,$$

and since $e^{-\delta x}$ is continuous on $[0,\infty)$, we conclude that there are constants $C, C(\delta)$ such that

$$\begin{aligned}e^{-\delta x} &\leqslant \min\left\{1, \frac{C}{(\delta x)^2}\right\}\\ &\leqslant C(\delta)\min\left\{1,\frac{1}{x^2}\right\}\\ &= C(\delta)\left(\mathbb{1}_{(0,1)}(x) + \mathbb{1}_{[1,\infty)}\frac{1}{x^2}\right)\end{aligned}$$

but the latter is an integrable function on $(0,\infty)$.

Problem 12.12 Without loss of generality we may assume that $\alpha\leqslant\beta$. We distinguish between the case $x\in(0,1)$ and $x\in[1,\infty)$. If $x\leqslant 1$, then

$$\frac{1}{x^\alpha} \geqslant \frac{1}{x^\alpha + x^\beta} \geqslant \frac{1}{x^\alpha + x^\alpha} = \frac{1/2}{x^\alpha + x^\alpha} \quad \forall\, x\leqslant 1;$$

this shows that $(x^\alpha + x^\beta)^{-1}$ is in $\mathcal{L}^1((0,1),dx)$ if, and only if, $\alpha < 1$.

Similarly, if $x \geqslant 1$, then

$$\frac{1}{x^\beta} \geqslant \frac{1}{x^\alpha + x^\beta} \geqslant \frac{1}{x^\beta + x^\beta} = \frac{1/2}{x^\beta + x^\beta} \quad \forall\, x \geqslant 1$$

this shows that $(x^\alpha + x^\beta)^{-1}$ is in $\mathcal{L}^1((1,\infty),dx)$ if, and only if, $\beta > 1$.

Thus, $(x^\alpha + x^\beta)^{-1}$ is in $\mathcal{L}^1(\mathbb{R},dx)$ if, and only if, both $\alpha < 1$ and $\beta > 1$.

Problem 12.13 If we use $X = \{1,2,\dots,n\}$, $x(j) = x_j$, $\mu = \epsilon_1 + \dots + \epsilon_n$ we have

$$\left(\sum_{j=1}^n |x_j|^p\right)^{1/p} = \|x\|_{L^p(\mu)}$$

and it is clear that this is a norm for $p \geqslant 1$ and, in view of Problem 12.18 it is not a norm for $p < 1$ since the triangle (Minkowski) inequality fails. (This could also be shown by a direct counterexample.

Problem 12.14 Without loss of generality we can restrict ourselves to positive functions—else we would consider positive and negative parts. Separability can obviously considered separately!

Assume that $\mathcal{L}^1_+$ is separable and choose $u \in \mathcal{L}^p_+$. Then $u^p \in \mathcal{L}^1$ and, because of separability, there is a sequence $(f_n)_n \subset \mathcal{D}_1 \subset \mathcal{L}^1$ such that

$$f_n \xrightarrow[n\to\infty]{\text{in } \mathcal{L}^1} u^p \implies u_n^p \xrightarrow[n\to\infty]{\text{in } \mathcal{L}^1} u^p$$

if we set $u_n := f_n^{1/p} \in \mathcal{L}^p$. In particular, $u_{n(k)}(x) \to u(x)$ almost everywhere for some subsequence and $\|u_{n(k)}\|_p \xrightarrow{k\to\infty} \|u\|_p$. Thus, Riesz' theorem 12.10 applies and proves that

$$\mathcal{L}^p \ni u_{n(k)} \xrightarrow[k\to\infty]{\text{in } \mathcal{L}^p} u.$$

Obviously the separating set $\mathcal{D}_p$ is essentially the same as $\mathcal{D}_1$, and we are done.

The converse is similar (note that we did not make any assumptions on $p \geqslant 1$ or $p < 1$—this is immaterial in the above argument).

Problem 12.15 We have seen in the lecture that, whenever $\lim_{n\to\infty} \|u - u_n\|_p = 0$, there is a subsequence $u_{n(k)}$ such that $\lim_{k\to\infty} u_{n(k)}(x) = u(x)$ almost everywhere. Since, by assumption, $\lim_{j\to\infty} u_j(x) = w(x)$ a.e., we have also that $\lim_{j\to\infty} u_{n(j)}(x) = w(x)$ a.e., hence $u(x) = w(x)$ almost everywhere.

Problem 12.16 We remark that $y \mapsto \log y$ is concave. Therefore, we can use Jensen's inequality for concave functions to get for the probability measure $\mu/\mu(X) = \mu(X)^{-1}\mathbb{1}_X\,\mu$

$$\begin{aligned}\int (\log u)\,\frac{d\mu}{\mu(X)} &\leqslant \log\left(\int u\,\frac{d\mu}{\mu(X)}\right)\\ &= \log\left(\frac{\int u\,d\mu}{\mu(X)}\right)\end{aligned}$$

$$= \log\left(\frac{1}{\mu(X)}\right),$$

and the claim follows.

Problem 12.17 As a matter of fact,

$$\int_{(0,1)} u(s)\,ds \cdot \int_{(0,1)} \log u(t)\,dt \leqslant \int_{(0,1)} u(x) \log u(x)\,dx.$$

We begin by proving the hint. $\log x \geqslant 0 \iff x \geqslant 1$. So,

$$\begin{aligned} &\forall y \geqslant 1 \,:\, \big(\log y \leqslant y \log y \iff 1 \leqslant y\big) \\ \text{and} \quad &\forall y \leqslant 1 \,:\, \big(\log y \leqslant y \log y \iff 1 \geqslant y\big). \end{aligned}$$

Assume now that $\int_{(0,1)} u(x)\,dx = 1$. Substituting in the above inequality $y = u(x)$ and integrating over $(0,1)$ yields

$$\int_{(0,1)} \log u(x)\,dx \leqslant \int_{(0,1)} u(x) \log u(x)\,dx.$$

Now assume that $\alpha = \int_{(0,1)} u(x)\,dx$. Then $\int_{(0,1)} u(x)/\alpha\,dx = 1$ and the above inequality gives

$$\int_{(0,1)} \log \frac{u(x)}{\alpha}\,dx \leqslant \int_{(0,1)} \frac{u(x)}{\alpha} \log \frac{u(x)}{\alpha}\,dx$$

which is equivalent to

$$\begin{aligned} &\int_{(0,1)} \log u(x)\,dx - \log\alpha \\ &\qquad = \int_{(0,1)} \log u(x)\,dx - \int_{(0,1)} \log\alpha\,dx \\ &\qquad = \int_{(0,1)} \log \frac{u(x)}{\alpha}\,dx \\ &\qquad \leqslant \int_{(0,1)} \frac{u(x)}{\alpha} \log \frac{u(x)}{\alpha}\,dx \\ &\qquad = \frac{1}{\alpha} \int_{(0,1)} u(x) \log \frac{u(x)}{\alpha}\,dx \\ &\qquad = \frac{1}{\alpha} \int_{(0,1)} u(x) \log u(x)\,dx - \frac{1}{\alpha} \int_{(0,1)} u(x) \log\alpha\,dx \\ &\qquad = \frac{1}{\alpha} \int_{(0,1)} u(x) \log u(x)\,dx - \frac{1}{\alpha} \int_{(0,1)} u(x)\,dx \log\alpha \\ &\qquad = \frac{1}{\alpha} \int_{(0,1)} u(x) \log u(x)\,dx - \log\alpha. \end{aligned}$$

The claim now follows by adding $\log\alpha$ on both sides and then multiplying by $\alpha = \int_{(0,1)} u(x)\,dx$.

Problem 12.18 Note the misprint: $q = p/(p-1) \iff \frac{1}{p} + \frac{1}{q} = 1$ independent of $p \in (1,\infty)$ or $p \in (0,1)$!

(i) Let $p \in (0,1)$ and pick the conjugate index $q := p/(p-1) < 0$. Moreover, $s := 1/p \in (1,\infty)$ and the conjugate index t, $\frac{1}{s} + \frac{1}{t} = 1$, is given by

$$t = \frac{s}{s-1} = \frac{\frac{1}{p}}{\frac{1}{p}-1} = \frac{1}{1-p} \in (1,\infty).$$

Thus, using the normal Hölder inequality for s,t we get

$$\begin{aligned}\int u^p \, d\mu &= \int u^p \frac{w^p}{w^p} \, d\mu \\ &\leqslant \left(\int \left(u^p w^p\right)^s \, d\mu\right)^{1/s} \left(\int w^{-pt} \, d\mu\right)^{1/t} \\ &= \left(\int uw \, d\mu\right)^{p} \left(\int w^{p/(p-1)} \, d\mu\right)^{1-p}.\end{aligned}$$

Taking pth roots on either side yields

$$\begin{aligned}\left(\int u^p \, d\mu\right)^{1/p} &\leqslant \left(\int uw \, d\mu\right) \left(\int w^{p/(p-1)} \, d\mu\right)^{(1-p)/p} \\ &= \left(\int uw \, d\mu\right) \left(\int w^{q} \, d\mu\right)^{-1/q}\end{aligned}$$

and the claim follows.

(ii) This 'reversed' Minkowski inequality follows from the 'reversed' Hölder inequality in exactly the same way as Minkowski's inequality follows from Hölder's inequality, cf. Corollary 12.4. To wit:

$$\begin{aligned}\int (u+v)^p \, d\mu &= \int (u+v)\cdot(u+v)^{p-1} \, d\mu \\ &= \int u\cdot(u+v)^{p-1} \, d\mu + \int v\cdot(u+v)^{p-1} \, d\mu \\ &\overset{\text{(i)}}{\geqslant} \|u\|_p \cdot \left\|(u+v)^{p-1}\right\|_q + \|v\|_p \cdot \left\|(u+v)^{p-1}\right\|_q.\end{aligned}$$

Dividing both sides by $\| |u+v|^{p-1} \|_q$ proves our claim since

$$\left\|(u+v)^{p-1}\right\|_q = \left(\int (u+v)^{(p-1)q} \, d\mu\right)^{1/q} = \left(\int (u+v)^{p} \, d\mu\right)^{1-1/p}.$$

Problem 12.19 By assumption, $|u| \leqslant \|u\|_\infty \leqslant C < \infty$ and $u \not\equiv 0$.

(i) We have

$$M_n = \int |u|^n \, d\mu \leqslant C^n \int d\mu = C^n \mu(X) \in (0,\infty).$$

Note that $M_n > 0$.

(ii) By the Cauchy-Schwarz-Inequality,

$$\begin{aligned}M_n &= \int |u|^n \, d\mu \\ &= \int |u|^{\frac{n+1}{2}} |u|^{\frac{n-1}{2}} \, d\mu\end{aligned}$$

$$\begin{aligned} &\leqslant \left(\int |u|^{n+1}\,d\mu\right)^{1/2}\left(\int |u|^{n-1}\,d\mu\right)^{1/2} \\ &= \sqrt{M_{n+1}M_{n-1}}\,. \end{aligned}$$

(iii) The upper estimate follows from

$$M_{n+1} = \int |u|^{n+1}\,d\mu \leqslant \int |u|^n \cdot \|u\|_\infty\,d\mu = \|u\|_\infty M_n.$$

Set $P := \mu/\mu(X)$; the lower estimate is equivalent to

$$\begin{aligned} &\left(\int |u|^n \frac{d\mu}{\mu(X)}\right)^{1/n} \leqslant \frac{\int |u|^{n+1}\frac{d\mu}{\mu(X)}}{\int |u|^n \frac{d\mu}{\mu(X)}} \\ \iff &\left(\int |u|^n\,dP\right)^{1+1/n} \leqslant \int |u|^{n+1}\,dP \\ \iff &\left(\int |u|^n\,dP\right)^{(n+1)/n} \leqslant \int |u|^{n+1}\,dP \end{aligned}$$

and the last inequality follows easily from Jensen's inequality since P is a probability measure:

$$\left(\int |u|^n\,dP\right)^{(n+1)/n} \int |u|^{n\cdot\frac{n+1}{n}}\,dP = \int |u|^{n+1}\,dP.$$

(iv) Following the hint we get

$$\|u\|_n \geqslant \left(\mu\{u > \|u\|_\infty - \epsilon\}\right)^{1/n}\left(\|u\|_\infty - \epsilon\right) \xrightarrow[\epsilon\to 0]{n\to\infty} \|u\|_\infty,$$

i.e.

$$\liminf_{n\to\infty} \|u\|_n \geqslant \|u\|_\infty.$$

Combining this with the estimate from (iii), we get

$$\begin{aligned} \|u\|_\infty &\leqslant \liminf_{n\to\infty} \mu(X)^{-1/n}\|u\|_n \\ &\overset{\text{(iii)}}{\leqslant} \liminf_{n\to\infty} \frac{M_{n+1}}{M_n} \\ &\leqslant \limsup_{n\to\infty} \frac{M_{n+1}}{M_n} \\ &\leqslant \|u\|_\infty. \end{aligned}$$

Problem 12.20 The hint says it all.... Maybe, you have a look at the specimen solution of Problem 12.19, too.

Problem 12.21 Without loss of generality we may assume that $f \geqslant 0$. We use the following standard representation of f, see (8.7):

$$f = \sum_{j=0}^{N} \phi_j \mathbb{1}_{A_j}$$

with $0 = \phi_0 < \phi_1 < \ldots < \phi_N < \infty$ and mutually disjoint sets A_j. Clearly, $\{f \neq 0\} = A_1 \uplus \cdots \uplus A_N$.

Assume first that $f \in \mathcal{E} \cap \mathcal{L}^p(\mu)$. Then

$$\infty > \int f^p \, d\mu = \sum_{j=1}^N \phi_j^p \, \mu(A_j) \geqslant \sum_{j=1}^N \phi_1^p \, \mu(A_j) = \phi_1^p \, \mu(\{f \neq 0\});$$

thus $\mu(\{f \neq 0\}) < \infty$.

Conversely, if $\mu(\{f \neq 0\}) < \infty$, we get

$$\int f^p \, d\mu = \sum_{j=1}^N \phi_j^p \, \mu(A_j) \leqslant \sum_{j=1}^N \phi_N^p \, \mu(A_j) = \phi_N^p \, \mu(\{f \neq 0\}) < \infty.$$

Since this integrability criterion does not depend on $p \geqslant 1$, it is clear that $\mathcal{E}^+ \cap \mathcal{L}^p(\mu) = \mathcal{E}^+ \cap \mathcal{L}^1(\mu)$, and the rest follows since $\mathcal{E} = \mathcal{E}^+ - \mathcal{E}^+$.

Problem 12.22 (i) Note that $\Lambda(x) = x^{1/q}$ is concave—e.g. differentiate twice and show that it is negative—and using Jensen's inequality for positive $f, g \geqslant 0$ yields

$$\begin{aligned}
\int fg \, d\mu &= \int g f^{-p/q} \, \mathbb{1}_{\{f \neq 0\}} f^p \, d\mu \\
&\leqslant \int f^p \, d\mu \left(\frac{\int g^q f^{-p} \mathbb{1}_{\{f \neq 0\}} f^p \, d\mu}{\int f^p \, d\mu} \right)^{1/q} \\
&\leqslant \left(\int f^p \, d\mu \right)^{1-1/q} \left(\int g^q \, d\mu \right)^{1/q}
\end{aligned}$$

where we used $\mathbb{1}_{\{f \neq 0\}} \leqslant 1$ in the last step. Note that $fg \in \mathcal{L}^1$ follows from the fact that $\big(g^q f^{-p} \mathbb{1}_{\{f \neq 0\}}\big) f^p = g^q \in \mathcal{L}^1$.

(ii) The function $\Lambda(x) = (x^{1/p} + 1)^p$ has second derivative

$$\Lambda''(x) = \frac{1-p}{p} \left(1 + x^{-1/p}\right) x^{-1-1/p} \leqslant 0$$

showing that Λ is concave. Using Jensen's inequality gives for $f, g \geqslant 0$

$$\begin{aligned}
\int (f+g)^p \mathbb{1}_{\{f \neq 0\}} \, d\mu &= \int \left(\frac{g}{f} \mathbb{1}_{\{f \neq 0\}} + 1 \right)^p f^p \mathbb{1}_{\{f \neq 0\}} \, d\mu \\
&\leqslant \int_{\{f \neq 0\}} f^p \, d\mu \left[\left(\frac{\int g^p \mathbb{1}_{\{f \neq 0\}} \, d\mu}{\int_{\{f \neq 0\}} f^p \, d\mu} \right)^{1/p} + 1 \right]^p \\
&= \left[\left(\int_{\{f \neq 0\}} g^p \, d\mu \right)^{1/p} + \left(\int_{\{f \neq 0\}} f^p \, d\mu \right)^{1/p} \right]^p.
\end{aligned}$$

Adding on both sides $\int_{\{f=0\}} (f+g)^p \, d\mu = \int_{\{f=0\}} g^p \, d\mu$ yields, because of the elementary inequality $A^p + B^p \leqslant (A+B)^p$, $A, B \geqslant 0$, $p \geqslant 1$,

$$\begin{aligned}
&\int (f+g)^p \, d\mu \\
&\leqslant \left[\left(\int_{\{f \neq 0\}} g^p \, d\mu \right)^{1/p} + \left(\int_{\{f \neq 0\}} f^p \, d\mu \right)^{1/p} \right]^p + \left[\int_{\{f=0\}} g^p \, d\mu \right]^{p/p} \\
&\leqslant \left[\left(\int g^p \, d\mu \right)^{1/p} + \left(\int f^p \, d\mu \right)^{1/p} \right]^p.
\end{aligned}$$